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AN

ELEMENTARY TREATISE

ON

CONIC SECTIONS,

AND

ALGEBRAIC GEOMETRY;

WITH A NUMEROUS COLLECTION OF EASY EXAMPLES,
PROGRESSIVELY ARRANGED.

Eſpecially deſigned for the Uſe of Schools and Beginners.

BY

G. HALE PUCKLE, M.A.

ST JOHN'S COLLEGE, CAMBRIDGE, PRINCIPAL OF WINDERMERE COLLEGE.

SECOND EDITION, REVISED AND ENLARGED.

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PREFACE TO THE FIRST EDITION.

THE following pages are intended to supply a want, which has, I think, been very generally experienced by those engaged in Mathematical Teaching. The appearance of Mr Salmon's elegant treatise rendered obvious the advantages of his harmonious and consecutive arrangement of the subject, but, at the same time, it seemed both desirable and possible that it should be laid before the beginner in a somewhat easier form.

My endeavour, then, has been, while proceeding upon a similar plan, to write with special reference to those difficulties and misapprehensions which commonly beset the student when he commences. With this object in view, I have dwelt at considerable length upon the earlier part of the subject, and have introduced geometrical and numerical illustrations of the analysis.

The Examples appended to each section were originally collected for the use of my own classes, and are mostly of a very elementary description. Even the more difficult will generally be within the reach of a student who has thoroughly mastered the text.

The work will, I hope, be found to contain all that is required by the upper classes of schools, and by the generality of Undergraduate Students at the Universities, and will also serve as a preparation for such as may wish to study more extensive modern treatises.

As my object has been perspicuity rather than originality, I have adopted those proofs with which we are familiar, and which will be found in many existing works, except where a shorter or clearer method of treatment has suggested itself. I have, however, with few exceptions, preferred purely algebraic proofs, at a sacrifice sometimes of brevity and elegance, since the geometrical treatment of these properties is now usually considered a distinct branch of the subject, and is studied separately.

My principal acknowledgments are due to Mr Salmon's treatise; but the works of O'Brien, Walton, Biot, Garnier, Young, Hymers, Latham, and Wrigley, together with the Cambridge Examination Papers, have also furnished valuable material, and the kindness of one or two friends has relieved the book of many of its original defects.

G. HALE PUCKLE.

ROYAL INSTITUTION SCHOOL, LIVERPOOL.
Jan. 1854.

PREFACE TO THE SECOND EDITION.

THE work has been thoroughly revised for the Second Edition. Explanatory matter has been added in those places where I found my own pupils at fault, and a few Articles have been introduced, which seemed necessary to meet the requirements of Examination Papers lately given in the Universities. My own experience has been decidedly in favour of the arrangement adopted in these pages; but those, who are of opinion that it is better to postpone the discussion of the General Equation of the Second Degree, may observe that a very little explanation will enable the Student to omit Section VII. when the subject is read for the first time.

I gladly take this opportunity of acknowledging that the First Edition owed much of its success to the valuable hints that I received from my friend, C. T. Hudson, M.A., of St John's College, Cambridge, Head Master of Bristol Grammar School.

WINDERMERE.
April, 1856.

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CORRECTIONS.

Page 37, line 2, *for* $= h$, *read* $h =$
„ 42, last line, *for* $=$, *read* $t =$
„ 43, line 2, *for* $m \mp mt$, *read* $1 \mp mt$
„ 58, line 18, *for* $B^2 + 4AC$, *read* $B^2 - 4AC$
„ 66, *for* depress, *read* raise, *and vice versâ*
„ 77, line 19, *for* $\left(\frac{A^2}{4} + \frac{B^2}{4} - C\right)$, *read* $\left(\frac{A^2}{4} + \frac{B^2}{4} - C\right)^{\frac{1}{2}}$
„ 92, line 6, *for* length, *read* square of length
„ 105, line 6, *for* AC, *read* $4AC$
„ 115, line 2, *for* P, *read* y
„ 148, line 4, *for* ρ, *read* ρ^2
„ 152, line 2, *for* $+ a^2$, *read* $- a^2$
„ 179, line 3, *dele* ‘equilateral.’
„ 237, line 8, *for* transverse, *read* major
„ 256, line 5, *for* $y_3 - y_2$, *read* $x_3 - x_2$, and *vice versâ*
„ 257, last line but one, *for* $\frac{b}{m}$, *read* $-\frac{b}{m}$
„ 262, last line, *for* $a^2 - b^2$, *read* $a^2 + b^2$

ALGEBRAIC GEOMETRY.

SECTION I.

Position of a point on a plane.—Loci.—Equations.

1. The first and most obvious purpose for which Algebra may be applied to Geometry is the investigation of problems which concern the *magnitude* of lines or areas. The ordinary operations of Trigonometry, in which lines are represented by algebraic symbols, will have already rendered the reader familiar with such investigations.

It is not, however, to problems of this nature that we are now about to direct our attention. It was shewn by Descartes, that Algebra might be applied to express the *position* of points, and the form of curves and geometrical figures. It is upon this branch of the subject that we are now about to enter, and we shall confine ourselves to such points, lines, &c. as may be supposed to lie in one plane, viz. that of the paper.

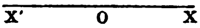
2. In ordinary Algebra, we have been in the habit of considering the symbols + and − as symbols of the reverse operations of addition and subtraction. But we may conceive of them in a much more general sense, as symbols of *exactly opposite affections*, and the use of them in elementary algebra will not be an exception to this definition, but a particular instance of it, inasmuch as a quantity to be added is exactly opposite in quality to one which is to be subtracted. We might easily multiply instances, in which it would be convenient to denote, by such symbols, opposite qualities which we have frequent occasion to express. Thus, let a man's property be a and his debts b; then we should say that the man is actually worth $+a-b$. If a be greater than b, the

man is solvent. If $a = b$, he is worth nothing at all, and if a be less than b, he is in debt. Hence, if we denote by p the man's actual property, we have

$$p = + a - b,$$

and the state of his affairs depends upon the magnitude and sign of p.

In Algebraic Geometry, we use these symbols to indicate *contrariety of position*. Thus, in any indefinite straight line, let us consider O as a fixed point from which distances are to be measured, and let us take two points X, X', equidistant from O and on opposite sides of it; then, if we denote the distance OX by $+ a$, we shall express the distance OX' by $- a$, *i. e.* we shall consider lines measured from left to right as positive, and lines measured from right to left as negative.

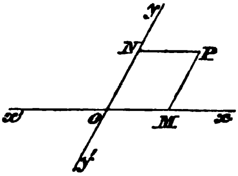


The propriety of this convention will best appear from an example. Let us suppose X the east and X' the west of O; then, if a man starting from O walk 6 miles to the east and then 2 miles to the west, the *magnitude* of his walk will be represented by 8, but his *position* with reference to his starting place by $+ 4$, or, in order to find his position, if we consider the distance he has walked east as positive, we must consider the distance he has walked west as negative. Similarly, if he walked 4 miles east and then 6 miles west, his *position* would be expressed by $- 2$, or two miles to the left of O, but the *magnitude* of his walk by $+ 10$. And, generally, if he walk a miles to the east and then b miles to the west, and we denote his position by p, we shall have

$$p = + a - b,$$

and his position with regard to O will be dependent on the magnitude and sign of p, but the length of his walk will be $a + b$. This distinction must be carefully borne in mind, and it must be remembered that the convention established above has reference, simply, to the *position* of points on the line XX', *with reference to* O; hence if we take $OX = +4$ and $OX' = -4$, the student must be careful to avoid the error of assuming XX' to be represented by $4 - 4$ or 0, as this would be reasoning about the magnitude of a line upon assumptions which have only been made about the position of its extremities with regard to a fixed point.

3. *To determine the position of a point in a plane.*



Let us suppose that we know the position of two straight lines Ox and Oy; then, if we draw the parallels PM, PN to these lines from any point P, we shall know the position of that point if the lengths of PM and PN are given. For example, if $PN = a$, $PM = b$ are two equations given us, we can determine the position of the point P with regard to the lines Ox and Oy; for, if we measure OM $(= a)$ along Ox, and ON $(= b)$ along OY, and complete the parallelogram $OMPN$, P will be the point whose position we wished to determine.

4. The line PM is usually denoted by the letter y and is called the *ordinate** of the point; OM, which $= PN$, is denoted by the letter x and is called the *abscissa* of the point; the two lines are called the *co-ordinates* of P. The lines xx', yy' are called the *axes of co-ordinates*, and their point of intersection O is called the *origin*: xx' is called the *axis of* x, and yy' the *axis of* y.

The point P is said to be determined when the values of x and y are given, as by the two equations $x = a$, $y = b$; as, for example, if it were given that $x = 3$ feet, $y = 2$ feet, we should determine the point of which x and y are the co-ordinates by measuring 3 feet along Ox and 2 feet along Oy, and completing the parallelogram of which these two lines formed the adjacent sides. The corner of the parallelogram opposite to O would be the position of the point. The point whose position is defined by the equations $x = a$, $y = b$ is commonly spoken of as the point (ab). The axes are said to be rectangular or oblique, according as angle yOx is or is not a right angle.

---

* The lines PM, &c. drawn parallel to one another from a series of points, were called by Newton 'lineæ ordinatim applicatæ,' and the abscissæ OM, &c. were the distances cut off by these lines from a fixed line as Ox.

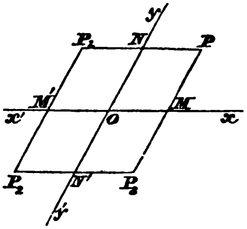
5. We have supposed hitherto that x and y, the co-ordinates of the point, are positive quantities, and have measured the distances along the lines Ox and Oy. If x or y be negative, it will indicate, according to the convention established above, that we must measure along Ox' or Oy', in order to find the position of the point. For example, if P, P_1, P_2, P_3 be points, situated in the four angles made by the axes, whose co-ordinates are of the same *magnitude*, *i. e.*

$$PM = P_1M' = P_2M' = P_3M = b,$$

and $OM = OM' = a$, these points will be represented by the following equations:

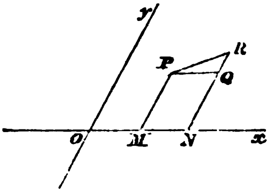
$$P \begin{cases} x = a, \\ y = b; \end{cases} \qquad P_1 \begin{cases} x = -a, \\ y = b; \end{cases}$$

$$P_2 \begin{cases} x = -a, \\ y = -b; \end{cases} \qquad P_3 \begin{cases} x = a, \\ y = -b. \end{cases}$$



The point represented by $x = 0$, $y = 0$ is the origin O; by $x = 0$, $y = b$ is the point N on the axis of y; by $x = a$, $y = 0$ is the point M on the axis of x, and so on.

6. *To find the distance between two points* P, R, *whose co-ordinates are known with reference to axes inclined to each other at a given angle.*



Let angle $yOx = \omega$, and let the co-ordinates of P be PM $(=y')$, OM $(=x')$, and of R, RN $(=y'')$, ON $(=x'')$;

draw PQ parallel to Ox, then

$$PQ = ON - OM = x'' - x',$$

$$RQ = RN - PM = y'' - y',$$

and $\angle\, PQR = \pi - \omega$; hence

$$PR^2 = PQ^2 + RQ^2 - 2PQ.RQ \cos PQR$$

$$= (x'' - x')^2 + (y'' - y')^2 + 2(x'' - x').(y'' - y') \cos \omega.$$

If one of the points as P were the origin, so that $x' = 0$, $y' = 0$, we should have

$$PR^2 = x''^2 + y''^2 + 2x''y'' \cos \omega.$$

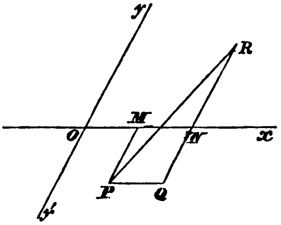
Cor. These formulæ become much more simple when $\omega = \frac{\pi}{2}$, or the axes are rectangular; in that case, since $\cos \omega = 0$,

$$PR^2 = (x'' - x')^2 + (y'' - y')^2,$$

or, if P be at the origin,

$$PR^2 = x''^2 + y''^2.$$

7. In using these formulæ, attention must be paid to the *signs* of the co-ordinates. If the point P, for instance, be in the angle xOy', the sign of its ordinate (y') will be negative, and we must write $y'' + y'$ instead of $y'' - y'$ in the formulæ; this may be seen to agree with the figure, as RQ will now $= RN + PM$.



The reader should draw figures, placing P and R in different compartments and in other varieties of position, that he may assure himself of the universal truth of the expressions obtained for the distance *PR*.

Ex. To find the distance between two points whose co-ordinates are $x = 2$, $y = -3$ and $x = -5$, $y = 6$, the axes being inclined at an angle of 60°.

Here $x'' - x' = -5 - 2 = -7$, $y'' - y' = 6 + 3 = 9$, and $\cos\omega = \frac{1}{2}$; hence, if d be the distance,

$$d^2 = 49 + 81 - 2 \cdot 7 \cdot 9 \cdot \tfrac{1}{2} = 67,$$

$$d = \sqrt{67}.$$

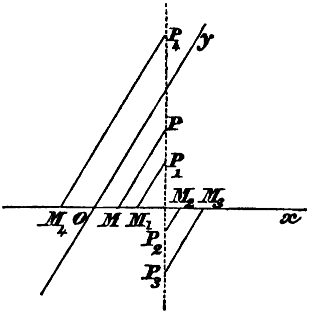
8. We may here remark that, in Algebraic Geometry, it is found that we need not trouble ourselves with any consideration of different *cases* of the same problem, as we did in the preceding Article, when we considered the cases where the two points were in the same angle yOx, and where they were in different angles yOx, $y'Ox$; for any general figure that we draw, consistent with the conditions of the problem, will lead us to a result that will be true for all possible cases comprehended in it, if we only pay attention to the *signs* of lines and angles, according to the rules laid down above, and interpret our results in accordance with our assumptions. The truth of the statement here made may be considered to be most satisfactorily proved by the perfect agreement of innumerable results, obtained by means of it, with those that might otherwise be obtained by the more laborious processes of common Geometry. An instance occurs in Euc. II. 13, of the different modifications of figure, &c. which must be introduced, in pure Geometry, to prove the same proposition.

9. We shall now shew how the method of co-ordinates may be applied to determine the position of straight lines and the form of curves, and we will begin with a few of the simplest cases.

We have seen that the position of a point P is completely determined by two equations $x = a$, $y = b$. Suppose we have *one* only of these equations, $x = a$, given us; then, evidently, if we draw a straight line PM parallel to Oy so that $OM = a$, the equation $x = a$ is satisfied by every point in that line produced every way indefinitely. Hence the equation $x = a$, instead of representing a point, represents a straight line parallel to the axis of y. In like manner the equation $y = b$ represents a straight line parallel to the axis of x. Hence the point $(a\,b)$ may be considered as the point of intersection of the two straight lines represented by the equations $x = a$, $y = b$, for at that point both equations are satisfied.

Again, the equation $x = 0$ is evidently satisfied by every point in the axis of y, and may therefore be said to represent that axis; and similarly, the equation $y = 0$ represents the axis of x. The intersection of these straight lines is the point where both the equations $x = 0$, $y = 0$ are satisfied, or the origin.

10. If, instead of two equations to determine x and y, we have a single equation, expressing a relation between them, (as, for instance, $2x + 3y + 9 = 0$,) the position of the point P will be indeterminate; for we may assign any values we please, OM, OM_1, OM_2, &c., to x, and the equation will furnish us with corresponding values of y, PM, P_1M_1, P_2M_2, &c., where we draw the ordinate downwards if the value which we assign to x gives us a negative value of y; hence we may obtain any number of points P, P_1, P_2, &c., whose co-ordinates satisfy the equation. We may take the points M, M_1, M_2, &c. as near as we please, and so get an assemblage of points P, P_1, P_2, &c. as near to one another as we please. If the points are



brought indefinitely near to one another, they will form a continuous line; hence we may consider the equation to represent the line which passes through *all* the points whose co-ordinates satisfy it.

Ex. Suppose the equation to be $y = 8 - 3x$; then, the unit of measure being known, if we take $OM = 1$, we have $PM = 5$, and P is a point in the line represented; if $OM_1 = 2$, $P_1 M_1 = 2$ and P_1 is another point; by taking values of x between 1 and 2 we may determine any number of points between P and P_1. If $OM_2 = 3$, $P_2 M_2 = -1$, and must therefore be drawn downwards; if $OM_3 = 4$, $P_3 M_3 = -4$, and must also be drawn downwards; by taking values of x between 2 and 3, or 3 and 4, we may obtain any number of points between P_1 and P_2 or P_2 and P_3. If we take a negative valve of x, or suppose $OM_4 = -1$, we have $P_4 M_4 = 11$, and so on: hence the line represented by the equation will be that which passes through P, P_1, &c. and all other points which satisfy the equation.

11. An equation between x and y may give us more than one value of y for each assumed value of x, or vice versâ; for example, the equation $y^2 = 6x$ will give, for every value of x, two equal values of y with opposite signs, so that, for every position of M, we shall have two positions of P, and two lines will be traced out at the same time.

When a point is restricted by conditions of any kind, to occupy any of a particular series of positions, that series of positions is called the *locus of the point*, or the *locus of the equation* that expresses the conditions, and the equation is called the *equation to the locus*, or the *equation to the curve* which passes through all the positions of the point.

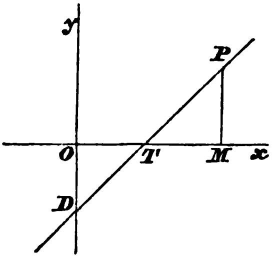
The curve then, or locus represented by an equation between x and y, is the assemblage of all those points whose co-ordinates satisfy the equation.

Hence if x', y' be quantities which, substituted for x and y, satisfy the equation, the point $(x'\ y')$ is a point on the locus.

12. The following simple examples will shew the method of representing loci by means of equations.

(i.) *To determine the locus of the equation* $y = x - 4$, *the axes being rectangular.*

Let Ox, Oy be the axes; now when $x = 0$ in the equation, $y = -4$; hence, if we take $OD = 4$ on the negative part of the axis of y, D will be a point in the locus (Art. 5). Again, when $y = 0$, $x = 4$; hence, if we take $OT = 4$, on the positive part of the axis of x, T will be a point in the locus. Draw the straight line DT, and produce it indefinitely both ways; this will be the locus of the equation; for take any point P in this line and let its co-ordinates be $OM = x$, $PM = y$; then, evidently, $PM = MT$, since the line makes an angle of 45° with Ox, and therefore



$$PM = OM - OT,$$
$$\text{or } y = x - 4.$$

Hence the co-ordinates of any point in the line DT satisfy the equation, and DT is the locus required.

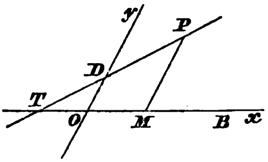
(ii.) *To determine the locus of the equation* $x^2 + y^2 = 9$, *the axes being rectangular.*

This equation asserts (Art. 6) that the distance of the point (xy) from the origin is constant and $= 3$. Hence the locus of the point is a circle whose centre is the origin and radius $= 3$.

13. As examples of determining the equation to a curve from some known property it possesses, we may take the following.

(i.) *To find the equation to the locus, of which every ordinate* (PM) *is greater than the corresponding abscissa* (OM) *by a given distance* (b).

The condition is evidently expressed by the equation $y = x + b$, and, if we take $OD = OT = b$, as in the figure, the line DT will plainly be the locus of the equation, for whatever be the position of P on the line, $PM = MT$, or $y = x + b$.



(ii.) *To find the equation to the locus, of which every ordinate* (PM) *is a mean proportional between* MO *and* MB, B *being a point in* Ox, *and the axes being rectangular.*

Let $OB = 2r$ and the angle yOx be supposed $= 90^0$; then since $PM^2 = OM \, . \, MB$, the equation is

$$y^2 = x\,(2r - x) \text{ or } y^2 = 2rx - x^2.$$

14. It is evident then that the examination of any equation between two unknown quantities must generally give rise to an assemblage of points, or a geometrical locus; and, conversely, if we can, by knowing some property of a curve or the circumstances of its description, determine a relation between the co-ordinates of any point taken arbitrarily on the curve, that relation is the equation to the curve.

If a problem relate to the position of a single point, and the data be sufficient to determine the position of that point, the problem is determinate; but, if one or more of the conditions be omitted, the data which remain may be sufficient to determine more than one point, each of which satisfies the conditions of the problem; the problem is then indeterminate. Such problems will in general result in indeterminate equations representing straight lines or curves, upon which the required points are situated. If, for example, the hypothenuse and one side of a right-angled triangle be given, the position of the vertex is fixed, for not more than one triangle can be described on the same side of the hypothenuse, with these data; but, if the hypothenuse only be given, the vertex may lie anywhere in the circumference of the semicircle described on the hypothenuse as diameter; the problem of finding the vertex is then indeterminate, and the semicircle is said to be the *locus* of all the vertices of all right-angled triangles described on one side of the given hypothenuse.

15. In the following pages we purpose to investigate those lines only which are represented by equations of the first and second degree between two variables (x and y). An equation of the first degree is an equation that involves no power of either of the variables higher than the first, nor their product; its most general form is

$$Ax + By + C = 0.$$

An equation of the second degree contains no term in which the sum of the indices of the variables is greater than two; its most general form is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

EXAMPLES I.

1. FIND the points whose co-ordinates are $(0, 1)$, $(-2, 1)$, $(-5, 0)$, $(-2, -3)$.

2. Draw a triangle the co-ordinates of whose angular points are $(0, 0)$, $(2, -3)$, $(-1, 0)$, and find the co-ordinates of the middle points of its sides.

3. A line cuts the positive part of the axis of y at a distance 4, and the negative part of the axis of x at a distance 3 from the origin: find the co-ordinates of the point where the part intercepted by the axes is cut in the ratio of $3 : 1$, the smaller segment being adjacent to the axis of x.

4. There are two points P $(7, 8)$ and Q $(4, 4)$: find the distance PQ, (i.) with rectangular axes, and (ii.) with axes inclined at an angle of 60°.

5. Work Ex. 4 when P is $(-2, 0)$, Q $(-5, -3)$.

SECTION II.

The straight line. General equation of the first degree.

16. *The general equation of the first degree between two variable quantities* (x *and* y) *represents a straight line.*

Every equation of the first degree is included in the form

$$Ax + By + C = 0, \quad . \quad . \quad . \quad (A)$$

A, B, and C being quantities which do not involve x and y; they are often numbers, are always invariable for any particular equation, and are therefore called *constants*. Let now (x_1y_1), (x_2y_2), (x_3y_3) be three points in the line (whatever kind of line it may be) represented by equation (A), and let the abscissæ be in order of magnitude, and therefore, from (A), the ordinates also in the same order. Then, since the relation among the co-ordinates is true for every point of the line, it is true for these three, and we have

$$Ax_1 + By_1 + C = 0, \quad . \quad . \quad . \quad (1)$$
$$Ax_2 + By_2 + C = 0, \quad . \quad . \quad . \quad (2)$$
$$Ax_3 + By_3 + C = 0, \quad . \quad . \quad . \quad (3)$$

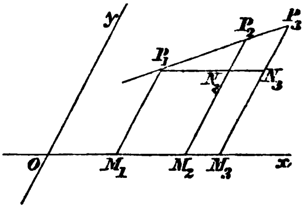
$$\text{from } (2) - (1) \quad A(x_2 - x_1) + B(y_2 - y_1) = 0,$$
$$\text{from } (3) - (1) \quad A(x_3 - x_1) + B(y_3 - y_1) = 0,$$

which evidently gives us

$$\frac{y_3 - y_1}{x_3 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now, if P_1, P_2, P_3 be the points (x_1y_1), (x_2y_2), (x_3y_3) and $P_1N_2N_3$ be drawn parallel to Ox, equation (4) gives us

$$\frac{P_3N_3}{P_1N_3} = \frac{P_2N_2}{P_1N_2}, \quad . \quad . \quad (5)$$



and, therefore, these three points are in the same straight line; for

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if not, let P_1P_3 cut P_2N_2 in P; then (Euc. VI. 2) we have

$$\frac{P_3N_3}{P_1N_3} = \frac{PN_2}{P_1N_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and, therefore, from (5) and (6), $P_2N_2 = PN_2$, which is absurd. In a similar manner it might be shewn that every other point in the *line* which the equation $Ax + By + C = 0$ represents, is in the same *straight line* with P_1 and P_2. This line is therefore a straight line.

17. It has been already stated that A, B, C are fixed quantities for any particular line. They are, therefore, the quantities which distinguish one line from another; for the same symbols x and y are used, not only in the same line for different points, but also for points in different lines. A little care and practice are sufficient to prevent this apparent ambiguity from causing any confusion.

For instance,

$$Ax + By + C = 0, \quad . \quad . \quad . \quad (1)$$

$$ax + by + c = 0, \quad . \quad . \quad . \quad (2)$$

represent (except when one equation is formed from the other and therefore expresses no new relation between x and y) two different straight lines, since A, B, C are different from a, b, c; but x and y, though the same in both, have not the same meaning in both, for, in the first case x and y represent the co-ordinates of any point in (1), and in the second the co-ordinates of any point in (2).

It is manifest that the position of the line does not depend upon the absolute magnitude of A, B, C, since, if we multiply or divide the equation by any constant, it will still represent the same line. It is seen, indeed, by dividing the equation by one of the constants A, B, C that there are in reality only two independent constants involved; for instance, if we divide by A, the equation becomes

$$x + \frac{B}{A}y + \frac{C}{A} = 0,$$

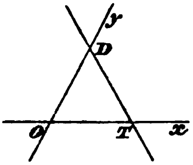
where $\frac{B}{A}$ and $\frac{C}{A}$ are the two constants that particularize the line; and we shall see hereafter that two conditions, which determine these constants, are sufficient and necessary to fix the position of a straight line in a plane.

We shall, hereafter, for the sake of brevity often speak of "the line represented by the equation $Ax + By + C = 0$," as "the line ($Ax + By + C = 0$)."

18. The converse of the preceding proposition is equally true, that all straight lines are represented by equations of the first degree; but if we wish to reason about any particular straight line, with reference to two fixed axes, we must have some data by means of which we may construct it, such as the position (with reference to the axes) of one or more points through which it passes, the angle it makes with either of the axes, its distance from the origin, &c., &c.; and we shall find that, as we suppose these data to vary, the form of the equation ($Ax + By + C = 0$) will vary too. We say *form*, because, as the line itself is supposed to remain the same, and only the means by which we determine its position to alter, it is plain that its equation must remain really the same also.

We shall now proceed to examine some of the most obvious of those conditions which fix the position of a line with reference to the axes, and find what form the equation will take in each particular case. We shall first deduce the forms from the general equation ($Ax + By + C = 0$), and afterwards verify our results by independent geometrical considerations.

19. (I.) Let DT be the line, and let the lengths DO and OT (which are called its intercepts on the axes) be the data to determine its equation. It is plain that these are sufficient to do so, as there can be only *one* straight line passing through D and T.



Let then $OD = b$, $OT = a$.

Now, in the equation

$$Ax + By + C = 0, \dots\dots\dots\dots\dots (A)$$

when $x = 0$, $y = -\frac{C}{B}$, which are the co-ordinates of the point D;

and

when $y = 0$, $x = -\frac{C}{A}$, $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$ T,

$$\text{and } \therefore -\frac{C}{B} = OD = b,$$

$$-\frac{C}{A} = OT = a;$$

but equation (A) may be written in form

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1, \dots\dots\dots (1)$$

and therefore becomes

$$\frac{x}{a} + \frac{y}{b} = 1,$$

for these data, and if by dividing out we write any equation of the first degree in the form of (1), the quantities in the denominators will be the intercepts on the axes.

20. (II.) Next let the data be the length of the perpendicular OE from the origin on DT, and the angle EOx which

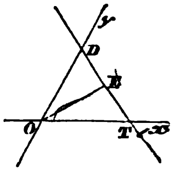
it makes with the axis of x. These being known, the point E is evidently known, and, as there can only be *one* line drawn through E at right angles to OE, these data are sufficient to determine the position of DT.

Let $OE = p$, angle $EOT = \alpha$, and $DOT = \omega$, the known angle between the axes.

Then, as before,

$$-\frac{C}{B} = OD, \text{ and } \therefore = \frac{p}{\cos(\omega - \alpha)},$$

$$-\frac{C}{A} = OT, \text{ and } \therefore = \frac{p}{\cos\alpha},$$



and therefore (A) becomes

$$\frac{x}{\dfrac{p}{\cos\alpha}} + \frac{y}{\dfrac{p}{\cos(\omega - \alpha)}} = 1,$$

$$\text{or} \quad x\cos\alpha + y\cos(\omega - \alpha) = p.$$

The coefficients of x and y in this equation are called the direction-cosines of the line. It must be carefully remembered that by α is meant the angle EOx which the perpendicular makes with the *positive* part of the axis of x, *i.e.* it is the angle through which Ox must be turned towards Oy, in order that Ox may coincide with OE, with x on the same side of O as E is; for instance, if the data to determine the position of the line were $OE = p$, angle $EOx = 180 + \alpha$, E would lie in EO produced, at a distance $= p$ from O; but p will in this case, and always, be a positive quantity, since the positive direction of OE may be defined as that which marks the boundary of the angle α.

Cor. 1. If $\omega = 90^0$, or the axes be rectangular, the equation becomes

$$x\cos\alpha + y\sin\alpha = p,$$

a very useful form of the equation.

Cor. 2. If the equation to any line be

$$Ax + By + C = 0, \text{.........}(1)$$

and the equation to the same line be written in the form

$$x \cos a + y \cos(\omega - a) = p, \text{.........}(2)$$

$$\text{or } x \cos a + y \sin a = p, \text{.........}(3)$$

according as the axes are oblique or rectangular, then, since equations (2) or (3) are really identical with (1), we see that A, B, and C are proportional to $\cos a$, $\cos(\omega - a)$, and $-p$ for oblique axes, and to $\cos a$, $\sin a$, and $-p$ for rectangular axes; and therefore, for the latter case,

$$\frac{A}{\cos a} = \frac{B}{\sin a} = -\frac{C}{p},$$

equations which are often useful. For example, if we wish to write equation (1) in the form (3), we have

$$\frac{\cos^2 a + \sin^2 a}{p^2} = \frac{A^2 + B^2}{C^2},$$

$$\text{or } \quad p = \frac{C}{\sqrt{A^2 + B^2}},$$

since p is always a positive quantity. Hence equation (1) must be written, supposing C a positive quantity,

$$-\frac{A}{\sqrt{A^2 + B^2}}x - \frac{B}{\sqrt{A^2 + B^2}}y = \frac{C}{\sqrt{A^2 + B^2}} \text{......} (4).$$

Hence, any equation may be written in the form of (3), by so adjusting the signs that the term not involving x or y may stand, as a positive quantity, as the right hand member, and then dividing the whole equation by the square root of the sum of the squares of the coefficients of x and y.

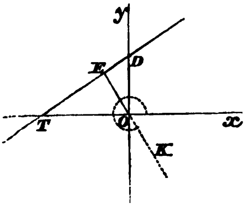
For instance, the equation

$$3x - 4y + 7 = 0$$

will be written

$$-\frac{3}{5}x+\frac{4}{5}y=\frac{7}{5}\ldots\ldots\ldots\ldots (5).$$

We say that p is always a positive quantity, because we have agreed above always to consider a as the angle which the perpendicular on the line, and not that perpendicular produced, makes with the positive part of the axis of x. If we were to remove this restriction, p might sometimes be a negative quantity, as we shall see hereafter (Art. 49), in the case of polar co-ordinates. For instance, equation (5) represents a line DT, where $\sin EOx=\frac{4}{5}$, $\cos EOx=-\frac{3}{5}$, and $OE=\frac{7}{5}$, but the equation may be written

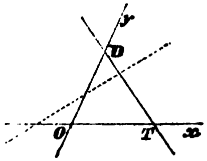


$$-\frac{4}{5}y+\frac{3}{5}x=-\frac{7}{5},$$

where we must take the angle KOx ($=180+EOx$) whose sine $=-\frac{4}{5}$, and cosine $=\frac{3}{5}$, and then measure $OE=\frac{7}{5}$ in what is now the negative direction of the perpendicular. This will evidently give us the same line DT; but in the following pages, when we speak of a line written in the above form, we shall always suppose p to be positive.

21. (III) Again, let the data be the length of OD and the angle DTx. Now, although we can draw an infinite number of lines through D, there is only one that makes this particular angle with the axis of x; for when we speak of the angle which a line makes with the axis of x, we always mean the angle which the part of the line above the axis makes with the axis produced in a positive direction, *i. e.* the angle

DTx, not *DTO*. These data then determine the line. Let $OD = b$, $DTx = a$, and $DOx = \omega$, as before.



$$\text{Then } -\frac{C}{B} = b$$

$$-\frac{C}{A} = OT = \frac{\sin(a-\omega)}{\sin a}\cdot b$$

$$= -\frac{\sin(\omega - a)}{\sin a}\cdot b,$$

$$\text{and } \therefore \frac{A}{B} = -\frac{\sin a}{\sin(\omega - a)};$$

$$\frac{-\frac{C}{B}}{-\frac{C}{A}} = +\frac{A}{B}$$

∠ C

but equation (A) may be written

$$y = -\frac{A}{B}x - \frac{C}{B},$$

and therefore the equation becomes for these data

$$y = \frac{\sin a}{\sin(\omega - a)}\cdot x + b,$$

which is often written

$$y = mx + b,$$

the constant m being a short way of writing the constant

$$\frac{\sin a}{\sin(\omega - a)}.$$

COR. 1. If $\omega = 90^0$, or the co-ordinates be rectangular,

$$m = \tan a.$$

Also m will be positive only when a is less than ω. In the figure the dotted line has m positive; in DT it is negative.

COR. 2. Every equation of the first degree may, by dividing by the coefficient of y and transposing, be written in the form

$$y = mx + b,$$

where, for rectangular axes, m represents $\tan\alpha$, and for oblique axes, m represents $\dfrac{\sin\alpha}{\sin(\omega-\alpha)}$, and where, for all axes, b represents the intercept of the line on the axis of y, and will be positive or negative according as the line cuts that axis above or below the origin.

Cor. 3. If the distance $OD = 0$, or the line pass through the origin, we shall have

$$-\frac{C}{B} = 0,$$

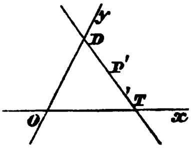
and the equation to the line will be

$$y = \frac{\sin\alpha}{\sin(\omega-\alpha)}\cdot x.$$

Hence the equation to any line passing through the origin is of the form

$$y = mx.$$

22. (IV) Next let the data be the point P' through which DT passes, and the angle DTx which it makes with Ox. Let the known co-ordinates of P' be x' and y', and let $\angle DTx = \alpha$, $DOx = \omega$, as before.



Then, since $(x'y')$ is a point on the line

$$Ax + By + C = 0,$$

$$\therefore\ Ax' + By' + C = 0,$$

and $\therefore\ A(x - x') + B(y - y') = 0,$

$$\text{or } \frac{y-y'}{x-x'} = -\frac{A}{B};$$

but as before,

$$-\frac{C}{B} = OD$$

$$-\frac{C}{A} = OT,$$

$$\text{and } \therefore \frac{A}{B} = \frac{OD}{OT} = -\frac{\sin \alpha}{\sin (\omega - \alpha)};$$

$$\text{hence } \frac{y - y'}{x' - x'} = \frac{\sin \alpha}{\sin (\omega - \alpha)},$$

$$\text{or } \frac{y - y'}{\sin \alpha} = \frac{x - x'}{\sin (\omega - \alpha)}.$$

Cor. 1. If $\omega = 90^0$, or the co-ordinates be rectangular,

$$\sin (\omega - \alpha) = \cos \alpha,$$

and the equation may be written

$$\frac{y - y'}{\sin \alpha} = \frac{(x - x')}{\cos \alpha},$$

or more commonly

$$y - y' = m (x - x'),$$

where $m = \tan \alpha$.

Cor. 2. If the given point be the origin and therefore $x' = 0$, $y' = 0$, this equation reduces to

$$y = mx,$$

as in III. Cor. 3.

23. (V) Next let the data be the co-ordinates of the two points P' and P'' through which the line passes. Let the co-ordinates of P' be x', y', and of P'', x'', y''; then since P' and P'' are points on the line

$$Ax + By + C = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore Ax' + By' + C = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{and } Ax'' + By'' + C = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\text{from (1) − (2)} \quad A (x - x') + B (y - y') = 0,$$

$$\text{from (3) − (2)} \quad A (x'' - x') + B (y'' - y') = 0,$$

whence we have for the required equation

$$\frac{y - y'}{x - x'} = \frac{y'' - y'}{x'' - x'}.$$

Cor. If one of the points $(x''y'')$ be the origin, the equation becomes, since $x''=0$, $y''=0$,

$$\frac{y-y'}{x-x'}=\frac{y'}{x'},$$

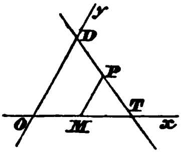
or, reducing,

$$y=\frac{y'}{x'}x$$

is the equation to a line passing through the origin and $(x'y')$.

24. All these equations may be obtained independently from geometrical considerations, instead of being deduced from the general equation of the first degree; thus

(I) The intercepts $OD=b$, $OT=a$ being given. Let $PM\ (=y)$, $OM\ (=x)$ be the co-ordinates of any point P in the line; then the triangles DOT, PMT are always similar, whatever be the position of P, and we have



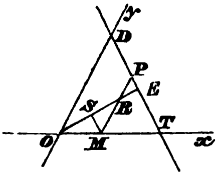
$$PM : MT :: OD : OT,$$

$$\text{or } \frac{y}{a-x}=\frac{b}{a},$$

$$\text{whence } \frac{x}{a}+\frac{y}{b}=1, \text{ as before.}$$

This equation will be easily remembered, as each variable stands over the intercept on its own axis. It is the same, whether the co-ordinates are rectangular or oblique.

25. (II) Given the perpendicular (p) and the angle (α) it makes with the axis of x. Let P be any point in DT; OM, PM (xy) its co-ordinates. Draw MS perpendicular to OE; then



$$\cos\alpha=\frac{OS}{x}\ \ldots\ldots(1)$$

$$\text{and}\quad \cos(\omega - a) = \cos SRM$$

$$= \frac{SR}{RM},$$

$$\text{and also} = \frac{RE}{RP},$$

$$\text{and} \therefore = \frac{SR + RE}{RM + RP},$$

$$= \frac{SE}{y} \ldots\ldots (2)$$

from (1) $OS = x \cos a$, from (2) $SE = y \cos(\omega - a)$;

$$\therefore\; x \cos a + y \cos(\omega - a) = OS + SE$$

$$= p.$$

Cor. If yOx be a right angle, we have

$$\cos a = \frac{OS}{x},\; \sin a = \cos SRM = \frac{SE}{y},$$

and we obtain the equation

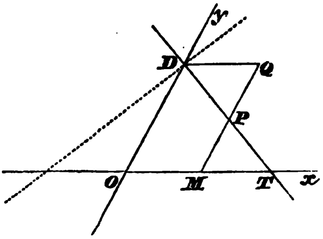
$$x \cos a + y \sin a = p.$$

26. (III) The length OD $(= b)$ and the angle DTx $(= a)$ being given.

Let OM $(= x)$ and PM $(= y)$ be the co-ordinates of any point P in the line; through D draw DQ parallel to Ox to meet MP produced in Q; then

$$\frac{PQ}{DQ} = \frac{\sin PDQ}{\sin DPQ},$$

$$\text{or}\quad \frac{QM - PM}{DQ} = \frac{\sin(180 - a)}{\sin(a - \omega)},$$



$$\text{or } \frac{b-y}{x} = -\frac{\sin a}{\sin(\omega - a)},$$

$$\text{whence } y = \frac{\sin a}{\sin(\omega - a)} . x + b,$$

$$\text{or} \quad y = mx + b,$$

as before.

Cor. If yOx be a right angle, we have

$$\frac{PQ}{DQ} = \tan PDQ,$$

$$= -\tan a,$$

$$\therefore \frac{b-y}{x} = -\tan a,$$

$$y = \tan a . x + b.$$

It will be observed that, in the figure we have chosen, m is a negative quantity, since $\sin(\omega - a)$ and $\tan a$ are both negative. If we take the dotted line, the geometrical construction will shew that m is positive in that case.

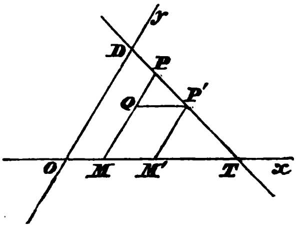
27. (IV) The co-ordinates $(x'y')$ of the point P', and the angle DTx $(= a)$ being given.

Let P be any point (xy) on the line, and draw $P'Q$ parallel to Ox, to meet the ordinate of P in Q;

$$\text{then } \frac{PQ}{P'Q} = \frac{\sin PP'Q}{\sin QPP'},$$

$$\text{or } \frac{y-y'}{x'-x} = \frac{\sin(180 - a)}{\sin(a - \omega)};$$

$$\text{hence } \frac{y-y'}{x-x'} = \frac{\sin a}{\sin(\omega - a)},$$



as before. This equation may be written

$$\frac{y-y'}{\sin\alpha} = \frac{x-x'}{\sin(\omega-\alpha)}, = \frac{PP'}{\sin\omega} = \frac{l}{\sin\alpha}$$

where each member of the equation is evidently $= \dfrac{P'P}{\sin\omega}$; hence we may write the equation thus $\dfrac{y-y'}{\sin\alpha} = \dfrac{y-x'}{\sin(\omega-\alpha)}$ $\sin\omega$ $\sin\omega$

$$\frac{y-y'}{s} = \frac{x-x'}{c} = l,$$

$$\text{where } s = \frac{\sin\alpha}{\sin\omega}, \quad c = \frac{\sin(\omega-\alpha)}{\sin\omega}, \quad l = P'P,$$

and where we must remember that s, c, are constants for any given line, but l is a variable quantity and equals the length of the line between the points (xy) and $(x'y')$.

Cor. If the axes be rectangular,

$$\frac{y-y'}{x-x'} = \tan\alpha,$$

and since $\sin\omega = 1$, the equation may be written

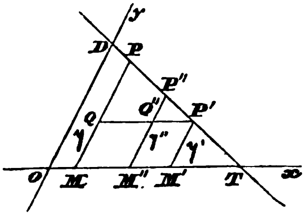
$$\frac{y-y'}{s} = \frac{x-x'}{c} = l,$$

where s and c are now equal to $\sin\alpha$ and $\cos\alpha$.

The geometrical meaning of this equation should be carefully noted, as we shall frequently have occasion to use it hereafter. The formula will evidently make l positive or negative, according as the distance is measured from P' towards D or T, and, as $\sin\alpha$ is always positive, the sign of l is always the same as that of $y-y'$.

28. (V) The co-ordinates of two points P' $(x'y')$ and P'' $(x''y'')$ being given.

Let P be any point (xy) on the line; and draw $P'Q''Q$ parallel to Ox, to meet the ordinates of P and P'' in Q and Q''; then



$$\frac{PQ}{QP'} = \frac{P''Q''}{Q''P'},$$

$$\text{or } \frac{y - y'}{x' - x} = \frac{y'' - y'}{x' - x''};$$

$$\text{or } \frac{y - y'}{x - x'} = \frac{y'' - y'}{x'' - x'},$$

as before.

This equation may be written in the form

$$y - y' = m\,(x - x'),$$

$$\text{where } m = \frac{y'' - y'}{x'' - x'}$$

$$= \frac{\sin a}{\sin (\omega - a)}, \text{ if the axes be oblique,}$$

$$= \tan a, \text{ if the axes be rectangular,}$$

which agrees with the result of (IV).

29. We will now give a few numerical examples on the preceding Articles.

Ex. 1. To find the equation to a line which cuts off intercepts on the axes of x and y equal to 3 and -5 respectively.

Writing $a = 3$, $b = -5$ in the equation of (I),

$$\frac{x}{a} + \frac{y}{b} = 1,$$

we have, for the required equation,

$$\frac{x}{3} - \frac{y}{5} = 1.$$

Ex. 2. The perpendicular from the origin on a straight line $= 5$, and makes an angle of 30^0 with the axis of x; find the equation to the line (i), when the axes are inclined at an angle of 60^0, and (ii), when they are rectangular.

Writing $\omega = 60^0$, $\alpha = 30^0$, and $p = 5$ in the equation of (II),

$$x \cos \alpha + y \cos (\omega - \alpha) = p,$$

we have

$$\sqrt{3}\,(x + y) = 10,$$

for the equation when the axes are inclined at an angle of 60^0; and if the axes are rectangular,

$$x \cos \alpha + y \sin \alpha = p,$$

or

$$\sqrt{3}\,x + y = 10,$$

is the equation required.

Ex. 3. To find the equation to a straight line which makes an angle of 135^0 with the axis of x, and cuts off an intercept $= -3$ on the axis of y, (i), if the co-ordinates are rectangular, and (ii), if they are inclined at an angle of 45^0.

(i) Writing $b = -3$, and $m = \tan 135^0 = -1$ in the equation of (III),

$$y = mx + b,$$

we have for the required equation,

$$y = -x - 3.$$

(ii) Writing $\alpha = 135^0$, $\omega - \alpha = 45^0 - 135^0 = -90^0$ in the equation of (III),

$$y = \frac{\sin \alpha}{\sin (\omega - \alpha)} \cdot x + b,$$

we have for the required equation,

$$y = \frac{\frac{1}{\sqrt{2}}}{-1} \cdot x - 3,$$

or
$$y = -\frac{1}{\sqrt{2}} \cdot x - 3.$$

Ex. 4. To find the equation to a straight line which passes through the points whose co-ordinates are

$$x = 4,\ y = -2, \text{ and } x = -3,\ y = -5,$$

writing 4 and -2 for x' and y', and -3 and -5 for x'' and y'' in the equation of (V),

$$\frac{y - y'}{x - x'} = \frac{y'' - y'}{x'' - x'},$$

we have, for the required equation,

$$\frac{y + 2}{x - 4} = \frac{-5 + 2}{-3 - 4},$$

or
$$7y - 3x + 26 = 0.$$

Ex. 5. To find the equation to a straight line which passes through the origin and a point whose co-ordinates are

$$x = 3,\ y = -2.$$

Writing 3 and -2 for x' and y' in the equation of (V) Cor.

$$y = \frac{y'}{x'} x,$$

we have

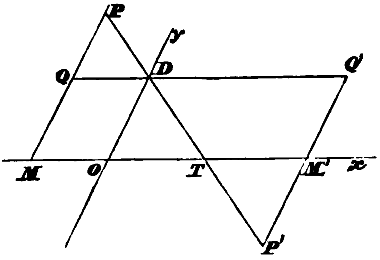
$$3y + 2x = 0,$$

for the equation required.

30. We have hitherto considered those points in the line only whose co-ordinates are positive, but it may be seen, in every case, by noticing the direction in which the co-ordinates

are drawn, that the equation is satisfied by every point in the line, if produced indefinitely both ways. We will take the form of equation given in (III), and suppose the equation to DT to be $y = mx + b$, where b, as we have shewn above, is a positive quantity and $= OD$, and m is, for this line, a negative quantity and

$$= \frac{\sin \alpha}{\sin (\omega - \alpha)}.$$



Hence, for the part between D and T, y is equal to the *difference* of the absolute lengths represented by b and mx, as was shewn in Art. 26; but if we take P on the other side of the axis of y, so that x is negative, the quantity mx becomes positive, and y is now equal to the *sum* of the absolute lengths represented by mx and b. We may easily see from geometrical considerations that this is the case, for PQ is the quantity which now represents the absolute length of the quantity mx or $\frac{\sin \alpha}{\sin (\omega - \alpha)} . DQ$ (the signs being disregarded) and $QM = b$; therefore PM or $y =$ the sum of PQ and QM or the sum of mx and b.

Again, if we take P' on the other side of the axis of x, the ordinate will, as in the portion DT, represent the difference of the absolute lengths of b and mx, since x is still positive and m is negative; and here $M'Q' = b$ and $P'Q' =$ the absolute length of mx or of $\frac{\sin \alpha}{\sin (\omega - \alpha)} DQ'$, and we see, geometrically, that the ordinate $P'M'$ is the difference of these lines: also it is drawn upwards or in the negative direction, as it evidently should be, since of the two quantities b and mx, whose difference it is, the negative quantity, mx, or $P'Q'$, is the greater.

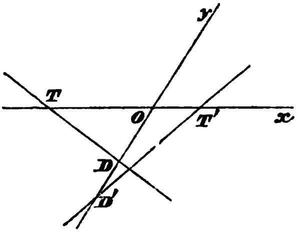
31. In order to trace the straight line represented by any equation of the first degree, it is only necessary to determine two points through which the line passes. If we make $x = 0$ in the equation, the value of y will give the point where the line cuts the axis of y, and if we make $y = 0$, the value of x will give the point where the line cuts the axis of x; the line joining these two points, and produced indefinitely both ways, will be the locus of the equation.

Ex. To trace the lines

$$\text{(i)} \quad 5y + 3x + 15 = 0.$$

$$\text{(ii)} \quad 2y - 5x + 10 = 0.$$

In (i), when $x = 0$, $y = -3$, or the line cuts the axis of y at a distance OD $(= 3)$ below the origin; when $y = 0$ $x = -5$, or the line cuts the axis of x at a distance OT $(= 5)$ to the left of the origin. Hence the line DT, produced indefinitely, is the locus of equation (i).



In (ii), when $x = 0$ $y = -5$, and

when $y = 0$ $x = 2$;

hence, if we take $OD' = 5$, and $OT' = 2$, $D'T'$ produced indefinitely will be the locus of equation (ii).

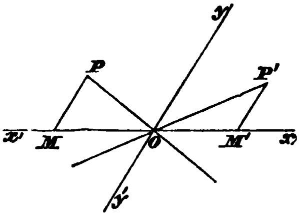
This method fails when the equation is of the form

$$Ax + By = 0;$$

for we have seen (Art. 21. Cor. 3), that in this case the line passes through the origin, since it may be written in the form $y = mx$. It will only be necessary then to determine one more point in the line, the method of doing which may be seen by the following examples.

To trace (i), $3x + 4y = 0$.

(ii), $2x - 3y = 0$.



In (i), $\frac{x}{y} = -\frac{4}{3}$, and the ordinate and abscissa must always have different signs, or the line will lie in the angles yOx', $y'Ox$; hence, if we take $OM = 4$, $PM = 3$, P is a point in the line, and PO produced indefinitely is the locus of equation (i).

In (ii), $\frac{x}{y} = \frac{3}{2}$, and the co-ordinates have the same sign, or the line lies in the angles yOx, $y'Ox'$; hence, if we take

$$OM' = 3, \; P'M' = 2,$$

P' is a point in the line, and the locus will be traced by joining OP' and producing the line indefinitely both ways.

very important

32. *The equations to two straight lines being given, to find the co-ordinates of their point of intersection.*

Since the co-ordinates of every point on a line must satisfy its equation, the co-ordinates of the point through which both the lines pass will satisfy both equations, or they will be the values obtained by solving the equations as simultaneous.

Ex. To find the co-ordinates of the point where the two lines

$$3y + 4x - 11 = 0, \; 4y + 3x - 10 = 0$$

intersect.

The values of x and y, obtained by combining these two equations, are $x = 2$, $y = 1$, which are, therefore, the co-ordinates of the point of intersection of the lines.

33. We remarked (Art. 19) that the variables x and y had not the same meaning in two different equations. If, however, we combine these equations in any manner whatsoever, we tacitly introduce the geometrical condition that the values of the variables *are* the same in each, and the result of such combination will, therefore, always express some fact regarding the point or points which the lines have in common. We may combine, as in Art. 32, in such a manner as absolutely to determine these co-ordinates, or, we may obtain some relation between them, expressed by an equation, and, as this equation must represent *some* line, straight or curved, and is satisfied by the co-ordinates of the point or points of intersection, it represents a line passing through those points.

We will take a simple example of this.

Let the equations to two straight lines be

$$y = mx + b \ldots (1), \; y = -\frac{1}{m}x + b' \ldots (2).$$

The geometrical meaning of these equations is (Art. 26) that the lines cut off intercepts on the axis of y equal to b and b', and that they make angles θ, θ' with the axis of x, so related that

$$\tan\theta \,.\, \tan\theta' = -1 \ldots\ldots (3).$$

Now, if θ and θ' be quantities whose value is absolutely known, we may treat (1) and (2) as simultaneous equations, and obtain definite values for x and y, the co-ordinates of the intersection of (1) and (2), in terms of the constants θ, θ', b, b'. But, if all we know about θ and θ' is that they are connected by equation (3), it is evident that for every value of θ there will be a corresponding value of θ', and that each pair will produce a new point of intersection. Suppose then that we combine (1) and (2) so as to eliminate this variable quantity

m, (which = $\tan\theta$, by Art. 26,) instead of one of the variables x or y. We obtain the equation

$$(y-b)(y-b')+x^2=0\ldots\ldots(4).$$

Equation (4) is a relation between the co-ordinates of the intersection of (1) and (2), and, as it represents a line of *some* kind, that line must pass through the intersection. But whatever value θ and θ' have, in (1) and (2), we shall, by the same means, obtain equation (4), provided only that the relation (3) exist between them. Hence, (4) passes through *every* point of intersection which can be generated by the change of θ, and is called the *locus of the intersection* of (1) and (2).

We may here remark that *the student cannot be too careful in considering the geometrical meaning of every algebraic step he takes.* The very facility with which algebraic expressions are combined and manipulated is often the most serious drawback to his progress.

34. In finding the co-ordinates of the points where two loci intersect, we shall sometimes fall upon impossible and sometimes upon infinite values of the variables. We gather from the former that the loci do not meet there, and the latter will often give us some information about the geometrical position of the loci. For instance, from the equations

$$y=mx+b\ldots(1),\quad y=m'x+b'\ldots(2),$$

we obtain, for the abscissa of the intersection,

$$x=\frac{b'-b}{m-m'}.$$

If now m becomes equal to m', x becomes infinite, from which we gather that (1) and (2) are parallel in this case, which is evidently true by Art. 26. The equation of the second degree will furnish us with other examples of this principle.

35. If the equations to two straight lines be

$$Ax+By+C=0\quad.\quad.\quad.\quad(1),$$
$$A'x+B'y+C'=0\quad.\quad.\quad.\quad(2),$$

the equation

$$(Ax + By + C) + k(A'x + B'y + C') = 0 \quad . \quad (3).$$

where k is any constant quantity, positive or negative, will represent a straight line passing through the point where (1) and (2) intersect; for it is an equation of the first degree, and therefore represents a straight line, and the co-ordinates of the point of intersection of (1) and (2) satisfy both those equations, and, therefore, evidently satisfy (3), or the point is on the line represented by (3).

Thus we see that the equations

$$(Ax + By + C) + 2(A'x + B'y + C') = 0,$$

$$(Ax + By + C) - 3(A'x + B'y + C') = 0,$$

represent two straight lines passing through the intersection of the lines,

$$Ax + By + C = 0, \; A'x + B'y + C' = 0.$$

Since k is an arbitrary quantity, undetermined by the nature of the problem, equation (3) will represent a *system* of lines fulfilling one condition only, viz. that of passing through the intersection of (1) and (2). *Any* individual of the system, particularized by the fulfilment of a second condition, can have the appropriate equation assigned to it by giving the proper value to k. For example, if any line of the system passed through a point $(x'y')$, we should obtain the value of k for that line from the equation

$$(Ax' + By' + C) + k(A'x' + B'y' + C') = 0,$$

and, substituting this value in (3), we should have the equation required. The student should exercise himself in obtaining k from other conditions, such as that the line (3) is parallel or perpendicular to a given line, &c.

Also, generally, if the equations to two curves be

$$S = 0, \; S' = 0,$$

where S and S' stand for expressions involving x and y, the equation

$$S + kS' = 0,$$

will represent a curve passing through the points of intersection of the two first-mentioned curves.

Hence, if any equation involve an arbitrary constant k in such a manner that the equation may be written in the form $S + kS' = 0$, that equation will represent a series of lines or curves which pass through the point of intersection of

$$S = 0,\ S' = 0.$$

Ex. 1. The equation

$$3y + 4x - 11 + k\,(4y + 3x - 10) = 0,$$

or

$$(3 + 4k)\,y + (4 + 3k)\,x - 11 - 10k = 0,$$

will, according as we give different values to k, represent a series of straight lines passing through the point ($x = 2$, $y = 1$,) which is the point of intersection of the two lines represented by

$$3y + 4x - 11 = 0,\ 4y + 3x - 10 = 0.$$

Ex. 2. The equation $y = mx + b$, may be written

$$(y - b) - mx = 0,$$

and, if m is indeterminate, this will represent a series of lines passing through the intersection of the lines

$$y - b = 0,\quad x = 0,$$

or cutting off an intercept $= b$ from the positive part of the axis of y, as we have already seen (Art. 21).

Ex. 3. The equation

$$y - y' - m\,(x - x') = 0,$$

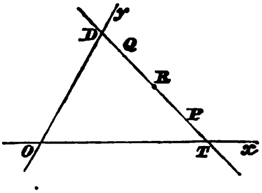
represents, if m is indeterminate, a system of lines passing through the intersection of the lines

$$y - y' = 0,\ x - x' = 0,$$

as we saw in Article 22.

36. *To find the co-ordinates of a point* (hk) *where the line joining two given points* ($x'y'$), ($x''y''$) *is cut in a given ratio.*

Let P, Q be the two points ($x'y'$), ($x''y''$), R the point (hk), whose co-ordinates are sought, ω the angle DOT, and let



$$PR : RQ :: m : n.$$

Now the equation to the line DT, which joins P and Q, since it passes through (hk), is

$$\frac{y-k}{s} = \frac{x-h}{c} = l \quad . \quad . \quad . \quad \text{(Art. 27)},$$

where $s = \dfrac{\sin \alpha}{\sin \omega}$, $c = \dfrac{\sin(\omega - \alpha)}{\sin \omega}$, α being the angle DTx, and where l is the distance of a point (xy) measured from R (hk), and is a positive quantity when measured towards D and negative when measured towards T. Hence, since ($x'y'$), ($x''y''$) are points in the line,

$$\frac{y'-k}{s} = \frac{x'-h}{c} = -RP,$$

and
$$\frac{y''-k}{s} = \frac{x''-h}{c} = RQ,$$

whence
$$\frac{y'-k}{y''-k} = -\frac{RP}{RQ} = -\frac{m}{n},$$

from which
$$k = \frac{my''+ny'}{m+n},$$

and similarly,
$$h = \frac{mx''+nx'}{m+n}.$$

If $m = n$, or the line be bisected in the point (hk),

$$= h\,\frac{x'' + x'}{2}, \quad k = \frac{y'' + y'}{2}.$$

EXAMPLES II.

1. Draw the lines whose equations are,

(1) $y = 5x + 2$, (2) $y - 7 = 5x + 3$, (3) $7y - 3x = 0$, Art. 31

(4) $6 - x = 2y$, (5) $\frac{x}{3} + \frac{y}{11} = 2$, (6) $2x + 3 = 0$.

2. Find the equation to the straight line which passes through the points $(2, 5)$ and $(0, -7)$. Art. 23

3. The co-ordinates of the angular points of a triangle being given, find the equations to the three lines each of which bisects two of the sides. Art 23

4. Two lines make each of them an angle of 45^0 with the axis of x, and their intercepts on the axis of y are 6 and 8; find the equation to the straight line which is equidistant from the two, the axes being rectangular. Art. 21

5. Find the equation to a straight line on which the perpendicular from the origin $= 6$, and makes (1) an angle of 45^0, and (2) an angle of 225^0 with the axis of x, the axes being rectangular. Art. [illegible]

6. Determine the point of intersection of the two lines $(3y - x = 0)$ and $(2x + y = 1)$. Art. [illegible]

7. The co-ordinates of P are $x = 2$, $y = 3$, and of Q, $x = 3$, $y = 4$; find the co-ordinates of R, so that $PR : RQ :: 3 : 4$.

8. Find the equation to a straight line which is equidistant from the two lines represented by the equation $y = mx + c \pm c'$.

9. The portion of a straight line intercepted by rectangular axes $= r$, and the perpendicular on it from the origin $= p$; what is the equation to the line.

10. Find the length of the perpendicular from the origin on the line $a(x-a)+b(y-b)=0$, and the portion intercepted by the axes, which are rectangular.

11. Find the area between the lines $y=x\tan\alpha$, $y=x\tan\alpha_1$, and $y=x\tan\alpha_2+b$, the axes being rectangular.

12. Find the equation to a straight line which passes through a given point (ab), and makes equal angles with the axes.

13. Find the equations to the diagonals of the parallelogram formed by the four lines $x=a$, $x=a'$, $y=b$, $y=b'$.

14. A straight line, inclined to the axis of x at an angle of 150^0, cuts the positive axes of rectangular co-ordinates in A and B; find the equation to a straight line bisecting AB and passing through the origin.

15. Find the equations to the four sides of a square, the co-ordinates of two of its opposite angular points being $(2, 3)$ and $(3, 4)$, the co-ordinates being rectangular.

16. Find the distance of the origin of co-ordinates from the line $\frac{x}{2}+\frac{y}{3}=1$, the axes being rectangular.

17. Find the equation to a straight line which passes through the intersection of the lines $x=a$, $x+y+a=0$, and through the origin.

18. The axes of co-ordinates being inclined to each other at an angle of 60^0, find the equation to a straight line parallel to the line $(x+y=3a)$, and a distance from it equal to $\frac{1}{2}a\sqrt{3}$.

19. Shew that the lines $y=2x+3$, $y=3x+4$, $y=4x+5$, all pass through one point.

20. Find the value of m in order that the line $(y=mx+3)$ may pass through the intersection of the lines $(y=x+1)$ and $(y=2x+2)$.

21. A straight line cuts off intercepts on the axes, the sum of the reciprocals of which is a constant quantity; shew that all straight lines which fulfil this condition pass through a fixed point.

22. A straight line slides along axes of x and y, and the difference of the intercepts is always proportional to the area it encloses; shew that the line always passes through a fixed point.

23. If the distance of a point from the origin = twice its distance from the axis of x, shew that it always lies in one of two straight lines that pass through the origin; axes rectangular.

24. Prove that the equation $x^2 + 2ax + y^2 + a^2 = 0$, represents a point in the axis of x; and $x^2 + 2ax + y^2 + 2a^2 = 0$ has no geometrical signification.

25. If a line cuts the (rectangular) axes of x and y at equal distances from the origin, and a line be drawn from the origin, dividing it in the ratio $m : n$, find the tangent of the angle which this latter line makes with the axis of x.

26. An equilateral triangle whose side $= a$ has its vertex at the origin, and its sides equally inclined to rectangular axes; find the co-ordinates of the angles, and thence of the point bisecting the base.

27. Shew that the three lines which bisect the angles of an equilateral triangle meet in a point, taking two of the sides as axes.

28. The base of a triangle and the line joining the bisection of the base with the vertex being axes, form the equations to the lines which join the bisection of the other sides with the opposite angles, and find their co-ordinates of intersection.

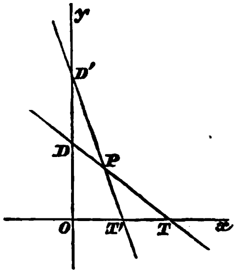
SECTION III.

The Straight Line continued.—Angles.—Perpendiculars.—Polar Co-ordinates.—Equations representing straight lines.

37. *To find the angle between two straight lines whose equations are given, the axes being supposed rectangular.*

Let tan $TPT' = t$, and let the equations to DT, $D'T'$ be

$$y = mx + b, \quad y = m'x + b',$$



since any equation may be written in this form; then (Art. 26)

$$m = \tan DTx, \quad m' = \tan D'T'x,$$

and $\tan TPT' = \tan(DTx - D'T'x)$,

Formula 61 of trigonometry

$$\text{or } t = \frac{m - m'}{1 + mm'},$$

which determines the angle TPT'.

Cor. Hence we see that, if $m = m'$, $\tan TPT' = 0$, or the lines are parallel. Also, if $m' = -\frac{1}{m}$, $\tan TPT' = \infty$, or the angle TPT' is a right angle.

The condition then that the two lines

$$y = mx + b, \quad y = m'x + b'$$

should be parallel, is

$$m = m',$$

and that they should be perpendicular to each other, is

$$m' = -\frac{1}{m}.$$

38. The condition of parallelism, deduced in the last Article, is evident independently, and we have, in fact, assumed it in previous Articles; for m and m' are the tangents of the angles which the lines make with the axis of x, and those angles are equal, if the lines be parallel. The condition of perpendicularity may be proved independently; for, if we consider TPT' a right angle, we have

$$m' = \tan D'T'x = -\cot DTx$$

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$$= -\frac{1}{m}.$$

39. The condition of parallelism may be easily seen to be the same if the co-ordinates be oblique, for, if each line make an angle a with the axis of x, we have, by Art. 26,

$$m = \frac{\sin a}{\sin(\omega - a)} = m'.$$

The condition of perpendicularity will be much more complicated when the co-ordinates are oblique than when they are rectangular, as, indeed, will all formulæ in which angles are involved. For this reason we shall choose rectangular axes in the solution of questions concerning angles, and shall not extend the remaining Articles of this Chapter to the case of oblique co-ordinates.

Ex. 1. To find the angle between the lines

$$2y + x + 1 = 0, \quad 3y - x - 1 = 0,$$

the axes being rectangular.

Here $m = -\frac{1}{2}$, $m' = \frac{1}{3}$, and, if a be the angle between the lines,

$$\tan a = \frac{-\frac{1}{2} - \frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}} = -1,$$

or $a = 135^\circ$.

Ex. 2. To find the angles between the lines

$$ax + by + c = 0,$$

$$a'x + b'y + c' = 0.$$

The equations may be written

$$y = -\frac{a}{b}x - \frac{c}{b},$$

$$y = -\frac{a'}{b'}x - \frac{c'}{b'};$$

$$\text{hence } m = -\frac{a}{b},\ m' = -\frac{a'}{b'},$$

$$\text{and } \tan \alpha = \frac{-\frac{a}{b} + \frac{a'}{b'}}{1 + \frac{aa'}{bb'}} = \frac{a'b - ab'}{bb' + aa'}.$$

40. *To find the equation to a straight line which shall make a given angle with a given straight line* ($y = mx + b$), *the axes being rectangular.*

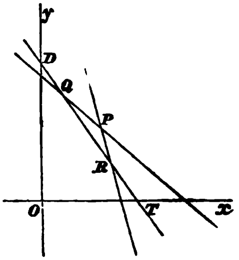
Let α be the given angle, and let $\tan \alpha = t$, and let the equation to the given line (DT) be

$$y = mx + b \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and the equation of the required line

$$y = m'x + b' \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where m' is to be found by the conditions of the problem.



Then, since the required line may lie either as PQ or PR, we shall have, by Art. 37,

$$t = \frac{m' - m}{1 + mm'}, \text{ or } = \frac{m - m'}{1 + mm'}.$$

Hence

$$m' = \frac{m' \pm t}{m \mp mt},$$

and the required equation is

$$y = \frac{m \pm t}{1 \mp mt} x + b' \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where b' still remains undetermined, as an infinite number of lines may be drawn fulfilling this condition.

If we add another condition, that the line should pass through a point $P(x'y')$, the equation will be (Art. 22, Cor. 1)

$$\begin{aligned} y - y' &= m'(x - x') \\ &= \frac{m \pm t}{1 \mp mt}(x - x') \quad . \quad . \quad . \quad . \quad . \quad . \quad (4), \end{aligned}$$

and it may be seen from the figure that there are, generally, two straight lines fulfilling these conditions.

Cor. 1. If $t = 0$, or the problem be *to find the equation to a straight line which passes through a given point and is parallel to a given straight line*, the equation is

$$y - y' = m(x - x').$$

Cor. 2. If $t = \infty$, or the problem be *to find the equation to a straight line which shall pass through a given point and be perpendicular to a given line*, the equation is

$$y - y' = -\frac{1}{m}(x - x'),$$

$$\text{for } \frac{m \pm t}{1 \mp mt} = \frac{\frac{m}{t} \pm 1}{\frac{1}{t} \mp m} = -\frac{1}{m}$$

if t becomes $= \infty$.

Ex. To find the equation to the straight lines which shall pass through the point (1, 2) and make an angle of 45° with the line

$$3x + 4y + 7 = 0.$$

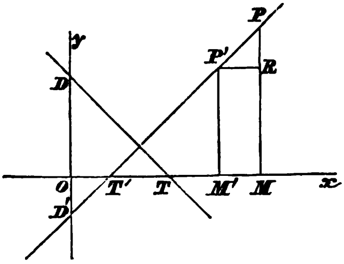
Here, $m = -\frac{3}{4}$, $t = 1$; hence, according as we take upper or lower signs in equation (4), we have

$$7y - x - 13 = 0, \quad y + 7x - 9 = 0,$$

for the required equations.

41. The equation obtained in Cor. 1, is the same as that of Art. 27, where its geometrical meaning is explained.

The geometrical meaning of the equation of Cor. 2, may be seen thus:



Let DT be the given line ($y = mx + b$), $P'(x'y')$ the given point, through which the line $D'T'$ is drawn perpendicular to DT, and let P be any other point (xy) on $D'T'$; then, if $P'R$ be drawn parallel to Ox to meet the ordinate PM in R, we have

$$\frac{PR}{P'R} = \tan PT'x = -\cot DTx,$$

or, since $\tan DTx = m$,

$$\frac{y - y'}{x - x'} = -\frac{1}{m}.$$

42. *To find the length of the perpendicular from a given point* $(x'y')$ *on the line* $(Ax + By + C = 0)$, *the axes being rectangular.*

The equation to a line passing through $(x'y')$ is (Art. 27)

$$\frac{y-y'}{s}=\frac{x-x'}{c}=l\,\ldots\ldots\ldots(1),$$

where l is the distance of any point (xy) of the line from the fixed point $(x'y')$, and if this line be perpendicular to $(Ax+By+C=0)$, we have (Art. 37)

$$Ax+By+C=0$$

$$By=-Ax-C \qquad y=-\frac{A}{B}x-\frac{C}{B}$$

$$\frac{s}{c}=\frac{B}{A}\,\ldots\ldots\ldots\ldots\ldots\ldots\,(2), \qquad y=mx+b$$

$$m=-\frac{A}{B}$$

since the two lines make with the axis of x angles whose tangents are, respectively, equal to

$$\frac{s}{c}\text{ and }-\frac{A}{B}.$$

If we substitute for x and y from (1) in the equation to the given line, the resulting equation will refer to the point where the two lines intersect, and we shall have for the distance (l) from $(x'y')$ to that point, *i. e.* for the perpendicular from $(x'y')$ on the given line, the equation

$$A(cl+x')+B(sl+y')+C=0,$$

$$\text{or}\quad (Ac+Bs)\,l+Ax'+By'+C=0\,\ldots\ldots\ldots\,(3).$$

But, since s and c are the sine and cosine of the angle which the line (1) makes with the axis of x,

$$s^2+c^2=1,$$

from which and from (2) we have

$$c=\pm\frac{A}{\sqrt{A^2+B^2}},\quad s=\pm\frac{B}{\sqrt{A^2+B^2}};$$

hence from (3)

$$l=\pm\frac{Ax'+By'+C}{\sqrt{A^2+B^2}},$$

according as we take c and s with the lower or upper signs.

As we are only now considering the *magnitude* of the perpendicular, the algebraic sign with which it is affected is immaterial. We shall see hereafter that it is sometimes necessary to select the appropriate sign for l; but, for this purpose, we shall use a geometrical construction, and the equation to the line in a less general form.

Cor. If the point $(x'y')$ be the origin, $x' = 0$, $y' = 0$, and we have, for the distance of the line from the origin,

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$$l = \frac{C}{\sqrt{(A^2 + B^2)}}.$$

$l = p$ in this Corollary.

Ex. To find the length of the perpendicular from the point $(x = 3,\ y = 5)$ on the line $(3y - 7x + 9 = 0)$.

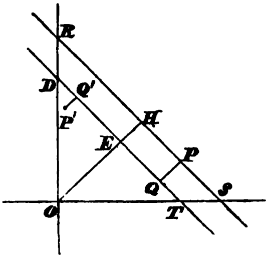
$$l = \frac{3 \times 5 - 7 \times 3 + 9}{\sqrt{7^2 + 3^2}} = \frac{3}{\sqrt{58}}.$$

43. *To find the length of the perpendicular from a point $(x'y')$ on the line $(x \cos \alpha + y \sin \alpha = p)$, the axes being rectangular.*

Let DT be the line

$$x \cos \alpha + y \sin \alpha = p,$$

and P the point $(x'y')$. Draw a line RS through P parallel to DT, and draw PQ, OEH perpendicular to the lines DT and RS; then the equation to RS must be



$$x \cos \alpha + y \sin \alpha = OH$$
$$= p' \text{ suppose};$$

but since $(x'y')$ is a point in RS, we have

$$x' \cos \alpha + y' \sin \alpha = p',$$

and $\therefore PQ = HE = p' - p$

$$= x' \cos \alpha + y' \sin \alpha - p,$$

and is therefore known.

If the point $(x'y')$ be on the other side of the line as P', the length of the perpendicular $P'Q'$ will evidently be $= p - p'$, or $= p - x' \cos a - y' \sin a$. Hence, we may deduce the following rule.

If the equation to a line be given in the form

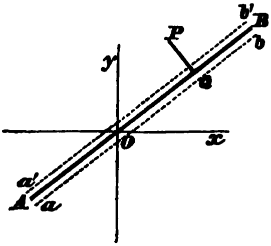
$$x \cos a + y \sin a = p,$$

where p is a positive quantity, the length of the perpendicular from $(x'y')$ is

$$\pm (x' \cos a + y' \sin a - p),$$

the upper or lower sign being used, according as we *do* or *do not*, pass through the line to reach the origin from the point $(x'y')$. If the line itself pass through the origin, the rule may be applied by considering the case of a new line, drawn parallel to the given line, at a very small distance from it, and making this distance vanish in the result.

44. An illustration will make the last remark of the preceding Article clearer.



Given a point $P(x'y')$, and a straight line AB $(Ax - By = 0)$, situated as in the fig.; to find the algebraical expression for PQ, which will make it a positive quantity.

The reasoning of Art. 43 is based upon the supposition that the equation to the line is written in a particular form, $(x \cos a + y \sin a = p)$, where p is a positive quantity. Now the equation to AB, when written in the required form is either

$$\frac{A}{\sqrt{A^2 + B^2}} x - \frac{B}{\sqrt{A^2 + B^2}} y = 0 \ldots\ldots\ldots\ldots (1),$$

or $$-\frac{A}{\sqrt{A^2 + B^2}} x + \frac{B}{\sqrt{A^2 + B^2}} y = 0 \ldots\ldots\ldots\ldots (2),$$

since, not having the sign of p to guide us, we may write it with equal accuracy in either form.

It will be necessary then to fix upon one of these forms, before we discuss the problem.

If we draw two lines ab, $a'b'$, parallel and very near to AB, it is evident that, for ab, $\cos\alpha$ is positive and $\sin\alpha$ negative, and the contrary for $a'b'$; so that the equation to AB will be written as (1) or (2) according as we consider it to be the limiting position of ab or $a'b'$, when the distance of these lines from AB becomes indefinitely small. Let us consider it as the limiting position of ab; then its equation is (1), and since we do *not* pass through the line to reach the origin,

$$PQ = \frac{B}{\sqrt{A^2+B^2}}\,y' - \frac{A}{\sqrt{A^2+B^2}}\,x'.$$

We shall evidently obtain exactly the same result if we consider AB to be the limit of $a'b'$, since then we *do* pass through the line to reach the origin.

This method may be readily applied to the bisectors of Art. 45, if one or both of them pass through the origin; for we may evidently reason about AB, as if it actually *were* ab, or $a'b'$, when we have written the equation in accordance with our assumption. We may remark that Arts. 43, 45, are equally true for oblique as for rectangular co-ordinates, if the form of the equation be modified as in Art. 20; but in that case the reduction of an equation to the required form becomes considerably more complicated.

45. It has already been shewn (Art. 35) that the equation

$$Ax + By + C + k(A'x + B'y + C') = 0 \ldots\ldots(1)$$

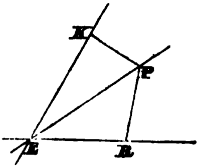
is the equation to a straight line passing through the intersection of the lines

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0 \ldots\ldots(2)$$

Now equation (1) admits of a very simple geometrical interpretation, if the lines (2) be written in the form

$$x\cos\alpha + y\sin\alpha - p = 0, \ldots\ldots(3)$$

$$x\cos\beta + y\sin\beta - q = 0; \ldots\ldots(4)$$



for let EK and ER be the lines (3) and (4), and suppose the origin somewhere in the angle opposite to KER. Then, if we take any point, as P, $(x'y')$, the perpendiculars PK and PR will be represented (Art. 43) by the expressions

$$x'\cos\alpha + y'\sin\alpha - p, \quad x'\cos\beta + y'\sin\beta - q,$$

and the equation

$$x'\cos\alpha + y'\sin\alpha - p - k(x'\cos\beta + y'\sin\beta - q) = 0 \ldots\ldots(5)$$

asserts that

$$\frac{PK}{PR} = \frac{k}{1},$$

and hence the locus of the point P, or of the equation

$$x\cos\alpha + y\sin\alpha - p - k(x\cos\beta + y\sin\beta - q) = 0 \ldots\ldots(6)$$

is a straight line passing through the intersection of (3) and (4), and such that, if perpendiculars be dropped, from any point in it, upon the lines (3) and (4), those perpendiculars will be to one another in the ratio of k to 1. Hence also, the sines of the angles which (6) makes with (3) and (4) are in the ratio of k to 1.

46. It will now be seen why we have (Art. 43) considered so carefully the *signs* of the perpendiculars, or, more correctly, investigated a method by which we may always be able to write the expressions for them, *so that those expressions shall represent positive quantities;* for if the lines in question be DT and $D'T'$, it is evident that the reasoning of

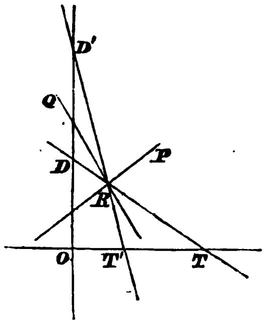
Article 43 will give us, for the point $P(x'y')$, if its perpendiculars on DT and $D'T'$ are as k to 1, the equation

$$\frac{x' \cos \alpha + y' \sin \alpha - p}{x' \cos \beta + y' \sin \beta - q} = \frac{k}{1},$$

or $x \cos \alpha + y \sin \alpha - p$

$$- k (x \cos \beta + y \sin \beta - q) = 0 \ldots (7)$$

as the equation to PR; whereas the point Q would, under the same circumstances, give us the equation



$$\frac{x' \cos \alpha + y' \sin \alpha - p}{-(x' \cos \beta + y' \sin \beta - q)} = \frac{k}{1},$$

or $x \cos \alpha + y \sin \alpha - p + k (x \cos \beta + y \sin \beta - q) = 0 \ldots (8)$
as the equation to QR.

The symbols α, β are often, for the sake of brevity, used for the expressions

$$x \cos \alpha + y \sin \alpha - p, \quad x \cos \beta + y \sin \beta - q,$$

and we may then write the equations (7) and (8)

$$\alpha - k\beta = 0, \quad \alpha + k\beta = 0.$$

47. In order to select the proper equation for any particular line passing through Q, where the ratio of the perpendiculars, or sines, is given, the following rule, which may be readily deduced from Art 43, is universally true.

If from any point in the required line, we have to pass through *both* or *neither* of the given lines to reach the origin, the equation is

$$\left.\begin{array}{l} \alpha - k\beta = 0, \\ \\ \alpha + k\beta = 0. \end{array}\right\} \ldots\ldots\ldots (9)$$

but if *only one*,

The equations to the lines, if given in any form, may (Art. 20, Cor. 2) be reduced to the form here used, and the above test applied.

Cor. If $k = 1$, or the perpendiculars are equal, equations (9) become

$$\alpha - \beta = 0, \quad \alpha + \beta = 0,$$

and represent the lines which *bisect* the angles $D'RT$ and $D'RD$.

48. Equations (6) and (7) (Arts. 45, 46) usually present some difficulty to the student, because we use the same symbols, x and y, to represent the co-ordinates of the point P and the co-ordinates of any point on the given lines. In these articles, we have endeavoured to avoid this difficulty, by first calling the co-ordinates of P x' and y', considering P as one fixed point. When we have obtained relations (5) between these co-ordinates, by means of the conditions of the problem, we may, evidently (Art. 17), write x and y for x' and y', in the equation to the *locus* of P, without any fear of confusion. We shall, however, hereafter, frequently speak of the perpendicular from the point (xy) on the line $x\cos\alpha + y\sin\alpha - p = 0$, bearing in mind this explanation.

Ex. *To find the equations to the lines which bisect the supplementary angles between the two lines.*

$$y - \sqrt{3}x - 5 = 0, \qquad \sqrt{3}y - x + 6\sqrt{3} = 0.$$

Let DR, $D'R$ be the lines, which will evidently lie as in the figure; then, comparing their equations with the equation

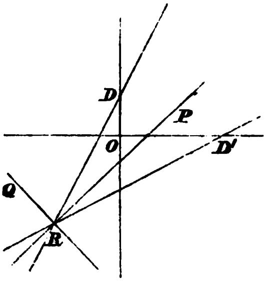
$$x\cos\alpha + y\sin\alpha = p,$$

and writing them in that form, we have (Art. 20, Cor. 2) for DR

$$-\frac{\sqrt{3}}{2}x + \tfrac{1}{2}y = \tfrac{5}{2},$$

and for $D'R$

$$\tfrac{1}{2}x - \frac{\sqrt{3}}{2}y = 3\sqrt{3}.$$



Hence the equation to the bisector *PR*, from any point of which we must pass through *neither* or *both* of the lines to reach the origin, is

$$\tfrac{1}{2}x - \frac{\sqrt{3}}{2}y - 3\sqrt{3} = -\frac{\sqrt{3}}{2}x + \tfrac{1}{2}y - \tfrac{5}{2},$$

$$\text{or } 2y - 2x = 11\sqrt{3} - 23.$$

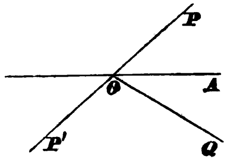
For the bisector *RQ*, where we pass over *one only*,

$$\tfrac{1}{2}x - \frac{\sqrt{3}}{2}y - 3\sqrt{3} = \frac{\sqrt{3}}{2}x - \tfrac{1}{2}y + \tfrac{5}{2},$$

$$\text{or } 2x + 2y = -23 - 11\sqrt{3}.$$

POLAR CO-ORDINATES.

49. Besides the method of expressing the position of a point that we have hitherto made use of, there is another which can often be employed with advantage. If a fixed point *O* be given, and a fixed line *OA* through it, we shall evidently know the position of any point *P*, if we know the length *OP* and the angle *POA*. The line *OP* is called the *radius vector*, the fixed point is called the *pole*, the line *OA* the *initial line;* and this method is called the method of *polar co-ordinates*. We shall, for the sake of brevity, call the point whose polar co-ordinates are ρ and θ, 'the point $(\rho\theta)$.'



50. The sign − is applied to polar co-ordinates on exactly the same principles as those already explained in the case of rectangular co-ordinates. Thus, if ρ represent any distance measured from *O* towards *P*, $-\rho$ represents an equal distance measured from *O* towards *P'*; and, if θ represent any angle

measured from A towards P, $-\theta$ will represent an equal angle measured from A towards Q. We shall define the *positive* direction of ρ as that part of the line which marks the boundary of the angle θ. A few examples will make this clearer.

Let a be any distance OP, measured from O towards P, θ being the angle which OP makes with OA (Art. 21); then,

Fig. 1.

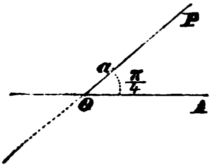


Fig. 2.

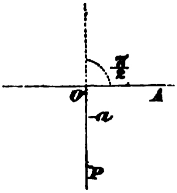


Fig. 3.

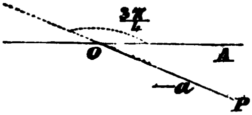


Fig. 4.

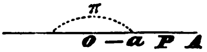
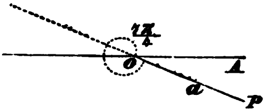


Fig. 5.



$$\theta = \frac{\pi}{4}, \quad \rho = a \quad \text{represents } P \text{ in Fig. 1;}$$

$$\theta = \frac{\pi}{2}, \quad \rho = -a \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{Fig. 2;}$$

$$\theta = \frac{3\pi}{4}, \quad \rho = -a \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{Fig. 3;}$$

$$\theta = \pi, \quad \rho = -a \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{Fig. 4;}$$

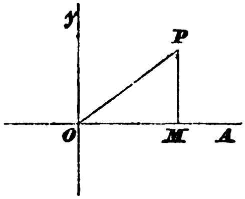
$$\theta = \frac{7\pi}{4}, \quad \rho = a \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{Fig. 5.}$$

It is important to observe that the direction in which ρ is measured depends not only on its sign, but also on the value of

θ; thus, when $\theta = \frac{3\pi}{4}$ and $\rho = -a$, ρ must be measured from O to P as in fig. 3; and when $\theta = \frac{7\pi}{4}$, $\rho = a$, ρ must be measured in exactly the same direction. Again, when $\theta = 0$, $\rho = a$, and when $\theta = \pi$, $\rho = -a$, ρ must in both cases be measured from O towards A.

51. *To transform polar co-ordinates into rectangular, or rectangular into polar.*

Suppose P to be the point whose polar co-ordinates OP $(= \rho)$, and angle POA $(= \theta)$ are known. Take the pole O as origin of rectangular co-ordinates, OA and a perpendicular through O, as axes of x and y, and let OM $(= x)$ and PM $(= y)$ be the rectangular co-ordinates of P.



Hence, we have

$$y = \rho \sin\theta, \qquad x = \rho\cos\theta,$$

$$x^2 + y^2 = \rho^2, \qquad \tan\theta = \frac{y}{x}.$$

These equations will enable us to transform any equation from polar co-ordinates to rectangular, and *vice versâ.*

Ex. 1. To find the general equation to the straight line, referred to polar co-ordinates.

Writing $x = \rho\cos\theta$, $y = \rho\sin\theta$, in the equation

$$Ax + By + C = 0,$$

we have

$$A\rho\cos\theta + B\rho\sin\theta + C = 0,$$

for the general polar equation.

Ex. 2. To transform the equation

$$a^2\rho \sin^2\theta = 2ab^2 \cos\theta - b^2\rho \cos^2\theta$$

to rectangular co-ordinates.

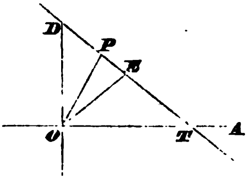
Multiplying by ρ, we have

$$a^2\rho^2 \sin^2\theta = 2ab^2\rho \cos\theta - b^2\rho^2 \cos^2\theta,$$

$$\text{or} \quad a^2y^2 = 2ab^2x - b^2x^2,$$

the initial line being now the axis of x, and the pole the origin.

52. It is easily seen that, as in the case of rectangular and oblique co-ordinates, the *form* of the general polar equation to the straight line will vary according to the data which determine the position of any particular line. The following data present the equation under its most convenient form.



Let DT be the line, and let the perpendicular $OE\ (=p)$ from the pole, and the angle $EOT\ (=\alpha)$ which it makes with the initial line, be the data to determine its position. Now if

$$A\rho\cos\theta + B\rho\sin\theta + C = 0, \quad . \quad . \quad (1)$$

be the equation to DT, we have, since $E(p\alpha)$ is a point on the line

$$Ap\cos\alpha + Bp\sin\alpha + C = 0. \quad . \quad . \quad (2)$$

From (1) and (2)

$$\frac{\rho\sin\theta - p\sin\alpha}{\rho\cos\theta - p\cos\alpha} = -\frac{A}{B}. \quad . \quad . \quad . \quad (3)$$

But from (1), when $\theta = 0$, $\rho = -\dfrac{C}{A}$, the point T,

and when $\theta = 90^0$, $\rho = -\dfrac{C}{B}$, ... D;

$$\text{and } \therefore -\frac{C}{B} = OD, \quad -\frac{C}{A} = OT,$$

$$\therefore \frac{A}{B} = \frac{OD}{OT} = \cot \alpha;$$

and, therefore, substituting in (3),

$$\rho \sin \theta . \sin \alpha - p \sin^2 \alpha = -\rho \cos \theta . \cos \alpha + p \cos^2 \alpha;$$

$$\therefore \quad \rho \cos (\theta - \alpha) = p,$$

$$\text{or} \quad \rho = p \sec (\theta - \alpha);$$

the polar equation for the above data.

Cor. This equation may be deduced at once from the equation to the straight line in the form

$$x \cos \alpha + y \sin \alpha = p,$$

by writing $\rho \cos \theta$ for x, and $\rho \sin \theta$ for y, the data being exactly the same in the two cases.

53. The equation of the preceding Article may be obtained very simply from geometrical considerations, for, if P be any point $(\rho\theta)$ in the line, we have

$$OP = OE . \sec POE,$$

$$\text{or} \quad \rho = p \sec (\theta - \alpha).$$

If $\alpha = 0$, or OE be taken for initial line, the equation is

$$\rho = p \sec \theta.$$

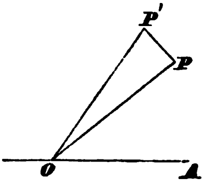
54. *To determine the distance between two points whose polar co-ordinates are given.*

Let P, P' be the two points, $OP = \rho$, angle $POA = \theta$, $OP' = \rho'$, angle $P'OA = \theta'$; then from the triangle POP', we have

$$PP'^2 = OP'^2 + OP^2 - 2\, OP . OP' \cos POP',$$

$$\text{or } PP'^2 = \rho'^2 + \rho^2 - 2\rho\rho' \cos(\theta' - \theta),$$

which gives the required distance.



EQUATIONS REPRESENTING STRAIGHT LINES.

55. If we have the equations to two straight lines

$$Ax + By + C = 0, \qquad A'x + B'y + C' = 0,$$

it is evident that the equation of the second degree

$$(Ax + By + C) . (A'x + B'y + C') = 0$$

will represent both the lines, for the co-ordinates of any point in either of the lines, substituted in the equation, will make one of the factors vanish, and the equation will be satisfied. Similarly, if we multiply together the equations to n straight lines, we shall obtain an equation of the nth degree which represents them all. Conversely, if any equation of the nth degree can be separated into factors of the first degree, it represents n straight lines.

56. *To find the condition that an equation of the second degree should represent two straight lines.*

Let the equation of the second degree, in its most general form,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

be written as a quadratic in y,

$$Cy^2 + (Bx + E)y + Ax^2 + Dx + F = 0,$$

$$\text{or } Py^2 + Qy + R = 0.$$

Solving this equation, we have

$$y = \frac{-Q \pm \sqrt{Q^2 - 4PR}}{2P};$$

where it is easy to see that Q can contain no higher power of x than the first, and $Q^2 - 4PR$ none higher than the second; for P does not contain x, and R no higher power of x than the second. Hence, if $Q^2 - 4PR$ contain x *at all*, it must be a perfect square, in order that the original equation may be resolved into two equations of the first degree.

If the quantity under the root do not contain x at all, the equation will represent two *parallel* straight lines; for, in this case, the coefficient of x in the right hand member of the equation, which is (Art. 21, Cor. 2) the tangent of the angle which the lines make with the axis of x, is not affected by the alteration of the sign of the radical.

Ex. 1. The equation

$$y^2 - x^2 = 0$$

when solved becomes

$$y = x, \text{ or } y = -x,$$

and represents (Art. 21, Cor. 3) two straight lines passing through the origin.

Ex. 2. The equation

$$Ax^2 + Bxy + Cy^2 = 0$$

can be resolved into the two equations

$$2Cy = -(B - \sqrt{B^2 - 4AC})x, \; 2Cy = -(B + \sqrt{B^2 + 4AC})x,$$

representing two straight lines passing through the origin.

Ex. 3. The equation

$$Ax^2 + Bx + C = 0$$

will give two solutions of the form $x =$ constant, and will therefore represent two straight lines parallel to the axis of y. In like manner the equation

$$Ay^2 + By + C$$

represents two straight lines parallel to the axis of x.

Ex. 4. Find the equations to the two lines represented by the equation

$$y^2 - 3xy + 2x^2 + 3y - 4x + 2 = 0.$$

This may be written

$$y^2 - 3(x - 1)y + 2(x - 1)^2 = 0;$$

solving for y, we have the equations

$$y - 2x + 2 = 0, \qquad y - x + 1 = 0,$$

which are the equations required, since the original equation is obtained by multiplying these two equations together.

57. *To find the angle between the lines represented by the equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad . \quad . \quad (1)$$

the axes being rectangular.

If (1) represents straight lines, it can be written in the form

$$(y - mx - b) . (y - m'x - b') = 0, \quad . \quad . \quad (2)$$

and the tangent of the angle between the two lines represented by (2) is (Art. 37)

$$= \frac{m - m'}{1 + mm'}.$$

But (2) may be written

$$mm'x^2 - (m + m')xy + y^2 + Px + Qy + R = 0, \quad . \quad . \quad (3)$$

where P, Q, and R do not involve x and y. Hence, if (1) and (3) are in reality the same equation

$$\frac{A}{mm'} = -\frac{B}{m + m'} = \frac{C}{1}, \quad . \quad . \quad (4)$$

$$\text{and } (m - m')^2 = (m + m')^2 - 4mm'$$

$$= \frac{B^2 - 4AC}{C^2} \quad . \quad . \quad \text{from (4)}$$

$$1 + mm' = \frac{A + C}{C};$$

$$\therefore \frac{m - m'}{1 + mm'} = \frac{(B^2 - 4AC)^{\frac{1}{2}}}{A + C}.$$

EXAMPLES III.

1. FIND the equation to the straight lines which pass through the point (1, 3) and make an angle of 30^0 with the line $(2y - x + 1 = 0)$; axes being rectangular.

2. Draw the lines represented by the equation

$$(2y - x + c) . (3y + x - c) = 0,$$

and determine (1) where they intersect, and (2) at what angle; the axes being rectangular.

3. Find the equation to a straight line which passes through the point $(c, 0)$ and makes an angle of 45^0 with the line $(bx - ay = ab)$; axes being rectangular.

4. Find the equation to a straight line which is perpendicular to the line $(8y + 5x - 3 = 0)$, and cuts the axis of y at a distance $= 8$ from the origin; axes being rectangular.

5. Find the cosine of the angle between the lines $(y - 4x + 8 = 0)$ and $(y - 6x + 9 = 0)$; axes being rectangular.

6. Find the angle between the lines $(4y + 3x + 5 = 0)$, and $(4x - 3y + 6 = 0)$; axes being rectangular.

7. Find the equations to the straight lines which pass through the intersection of the lines $(y = 2x + 4)$, $(y = 3x + 6)$, and bisect the supplementary angles between them; axes being rectangular.

8. What is the geometrical signification of the equations

$$x^2 + y^2 = 0, \qquad xy = 0?$$

9. Find the equations to the straight lines which bisect the angles between the lines $(5y - 2x = 0)$ and $(3y + 4x = 12)$; axes being rectangular.

10. The lines represented by the equation

$$6y^2 + xy - 2x + y - x^2 - 1 = 0,$$

are inclined to one another at an angle of 135^0; axes being rectangular.

11. The equation $2y^2 - 3xy - 2x^2 - 3y + 6x = 0$, represents two straight lines at right angles; axes being rectangular.

12. The equation $y^2 - 2xy \sec\theta + x^2 = 0$, represents two straight lines inclined to one another at an angle θ; axes being rectangular.

13. What is the inclination of the co-ordinate axes, when the lines represented by $y^2 - x^2 = 0$, are perpendicular to one another?

14. What lines are represented by the equation

$$\left(\frac{x}{a}+1\right)^2 + \frac{3}{2}\left(\frac{x}{a}+1\right)\frac{y}{b} = \frac{y^2}{b^2}\ ?$$

15. What must be the inclination of the axes in order that the lines $(xy - 3y - 2x + 6 = 0)$ may include an angle of 135^0?

16. Find the equations to the two straight lines which pass through the origin and divide into three equal parts the distance between the points in which the axes of co-ordinates are intersected by the line

$$(x + y = 1).$$

17. Find the distance of the point of intersection of the lines

$$(3x + 2y + 4 = 0), \quad (2x + 5y + 8 = 0),$$

from the line $(y = 5x + 6)$, the axes being rectangular.

18. The polar co-ordinates of P are $\rho = 5$, $\theta = 75^0$, and of Q $\rho = 4$, $\theta = 15^0$; find the distance PQ.

19. Find the polar co-ordinates of the point of intersection of the lines $\left\{\rho = 2a \sec\left(\theta - \frac{\pi}{2}\right)\right\}$ and $\left\{\rho = a \sec\left(\theta - \frac{\pi}{6}\right)\right\}$, and the angle between them.

20. Trace the line whose polar equation is

$$\rho = 2a \cos\left(\theta + \frac{\pi}{6}\right).$$

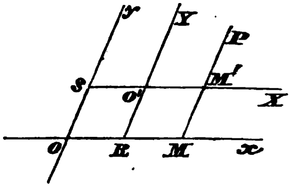
SECTION IV.

Transformation of Co-ordinates.

58. When the position of a point or the equation to a curve is given with reference to any particular system of co-ordinates, it is frequently necessary to find that position or equation with regard to some other system. We have already shewn how to pass from rectangular to polar co-ordinates, and the converse, and we shall now shew how we may perform other transformations, such as altering the origin, passing from rectangular to oblique co-ordinates, and others of the same nature.

59. *To transfer the origin of co-ordinates to a point* $(x'\ y')$ *without altering the direction of the axes.*

Let $O'X$, $O'Y$ be the new axes, respectively parallel to Ox and Oy, the old ones; let the co-ordinates of any point P, referred to the old axes, be x, y, and, when referred to the new, X, Y; then



$$OR = x',\ O'R = y', \text{ and}$$

$$PM = PM' + O'R,\ OM = O'M' + OR,$$

$$\text{or } y = Y + y',\ x = X + x'.$$

These formulæ are equally true for rectangular and oblique co-ordinates.

Hence, to find what the equation to any locus becomes, when the origin is transferred to a point $(x'\ y')$, the new axes remaining parallel to the old, we must write $x + x'$ for x, and $y + y'$ for y.

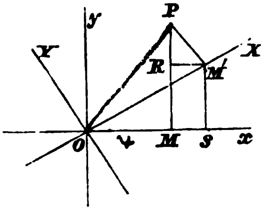
Ex. To find what the equation $y^2 + 4y - 4x + 8 = 0$ becomes when the origin is transferred to a point whose co-ordinates are $x = 1$, $y = -2$.

Writing $x+1$ for x, and $y-2$ for y, the equation becomes

$$y^2 = 4x.$$

60. *To find what the co-ordinates of a point become, if the axes, being rectangular, are turned through a given angle* (α), *the origin remaining the same.*

Let Ox, Oy, OM, PM be the old axes and co-ordinates of any point P, OX, OY, OM', PM' the new ones; let angle $XOx = \alpha$, and let the old co-ordinates be x, y, the new X, Y; then, transforming to polar co-ordinates (Art. 51) we have, if the angle $POx = \theta$,



$$x = \rho \cos\theta, \qquad y = \rho \sin\theta,$$
$$X = \rho \cos(\theta - \alpha), \qquad Y = \rho \sin(\theta - \alpha).$$

$$\begin{aligned} \text{Hence,} \quad x &= \rho(\cos\theta - \alpha + \alpha) \\ &= \rho\cos(\theta-\alpha)\,.\cos\alpha - \rho\sin(\theta-\alpha)\sin\alpha \\ &= X\cos\alpha - Y\sin\alpha. \end{aligned}$$

$$\begin{aligned} \text{Similarly,} \quad y &= \rho\sin(\theta - \alpha + \alpha) \\ &= \rho\sin(\theta-\alpha)\cos\alpha + \rho\cos(\theta-\alpha)\sin\alpha \\ &= Y\cos\alpha + X\sin\alpha. \end{aligned}$$

These results may be seen to agree with the figure; for, if $M'R$, $M'S$ be drawn parallel to Ox, Oy respectively, we have

$$OM = OS - RM',$$
$$PM = PR + M'S,$$

or, since angle RPM' = angle $XOx = \alpha$,

$$x = X\cos\alpha - Y\sin\alpha, \quad y = Y\cos\alpha + X\sin\alpha;$$

hence, to find what the equation to any locus becomes, when referred to the new axes, we must write

$$x\cos\alpha - y\sin\alpha \text{ for } x, \text{ and } y\cos\alpha + x\sin\alpha \text{ for } y.$$

Ex. To find what the equation $x^2 - y^2 = a^2$ becomes when the axes are moved through an angle of 45^0. Here,

$$\sin a = \cos a = \frac{1}{\sqrt{2}},$$

and we must write

$$\frac{1}{\sqrt{2}}(x - y) \text{ for } x, \text{ and } \frac{1}{\sqrt{2}}(x + y) \text{ for } y;$$

hence, the equation becomes

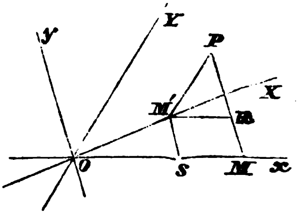
$$(x - y)^2 - (x + y)^2 = 2a^2,$$
$$\text{or } 2xy + a^2 = 0.$$

61. *To find what the co-ordinates of a point become when the axes are changed from one oblique system to another, the origin remaining the same.*

Using the same notation as in Art. 60 and a similar figure, we have

$$PM = M'S + PR$$
$$= OM' \frac{\sin (Xx)}{\sin (xy)} + PM' \frac{\sin (Yx)}{\sin (xy)};$$

where by sin (xy) we mean the sine of the angle which the axis of x makes with the axis of y, and by sin (Xx) the sine of the angle which the axis of X makes with the axis of x; and so with the rest.



Similarly, $OM = OS + M'R$

$$= OM' \frac{\sin (Xy)}{\sin (xy)} + PM' \frac{\sin (Yy)}{\sin (xy)}.$$

Hence, if we assume angle $(xy) = \omega$, angle $(Xx) = \alpha$, angle $(Yx) = \beta$, we have

$$y = \frac{X \sin \alpha}{\sin \omega} + \frac{Y \sin \beta}{\sin \omega},$$

$$x = \frac{X \sin (\omega - \alpha)}{\sin \omega} + \frac{Y \sin (\omega - \beta)}{\sin \omega}.$$

If these values of x and y be substituted in the equation to any curve referred to the old axes, we shall have the equation to the same curve referred to the new axes.

Cor. 1. If we wish at the same time to transfer the origin to a point $(x'y')$, we have only to add x' and y' to these expressions for x and y respectively.

Cor. 2. The formulæ of this Article include the particular cases where we wish to pass from rectangular to oblique, or from oblique to rectangular axes. In making use of these formulæ it must be borne in mind that ω represents the angle formed by the positive directions of the old axes, and that α, β represent the angles formed by the positive directions of the new axes with the positive direction of Ox.

62. The student must bear in mind that we make no change in the *locus*, by changing the origin or axes. The assemblage of points represented by the new equation is precisely the same as that represented by the old, but the manner in which the axes are placed with regard to them is changed, and therefore the equation, which expresses this *relative* position, is not the same as before.

We may notice also that the *degree* of an equation cannot be altered by transformation. For let Ax^py^q represent any term of an equation of the n^{th} degree, where $p + q = n$; then, since (Arts. 59—61) the old co-ordinates are functions of the new, of the first degree only, we should have

$$Ax^py^q = A\,(ax' + by' + c)^p\,(dx' + ey' + f)^q,$$

and no term resulting from this multiplication can be of a higher degree than $p+q$ or n. Hence, the equation cannot be ~~depressed~~ raised. It, consequently, cannot be ~~raised~~ depressed; for, if that were possible, we might, by re-transforming it, ~~depress~~ raise it, which has been proved to be impossible.

63. In the case of polar co-ordinates, if we wish to turn the initial line through an angle α, we must write $\theta+\alpha$ for θ in the equation, since the new θ is less than the old θ by the angle α.

Examples of the transformation of co-ordinates will be found at the end of Section VII.

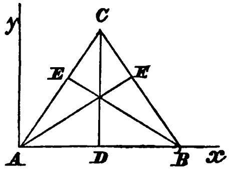
SECTION V.

Geometrical Applications.

64. We shall now give a few examples of the application of the formulæ we have obtained to the solution of geometrical problems. In attempting to solve these problems algebraically, the student will find that much depends upon a judicious selection of the origin and axes, and the application of the proper equations and formulæ. He should in every case consider the problem well before he attempts the solution, and form a *definite plan* before he begins. He may very possibly not be able to carry out his original scheme, but his attempts to do so will probably suggest some method by which he may solve the problem, and he will, at any rate, avoid a practice very common to beginners, of working without any definite aim, and, consequently, introducing and combining equations and formulæ that only serve to embarrass him, without in any way aiding him in the solution.

65. *To shew that the perpendiculars drawn from the vertices on the opposite sides of a triangle meet in a point.*

Let ABC be the triangle, CD, BE, AF the perpendiculars, and assume Ax, Ay as rectangular axes; let the co-ordinates of C be $AD = x'$, $CD = y'$, and let $AB = x''$. Now the proposition is proved, if we can shew that the abscissa of the point where AF and BE intersect is $= x'$, for they will then, evidently, intersect in CD. In order to shew this we must find their equations, which we shall obtain (Art. 40, Cor. 2) by observing that they each pass through a given point and are perpendicular to a given line. We must



then first find the equation to the lines *AC*, *BC* to which they are perpendicular.

Since *AC* passes through the origin and the point $C(x'y')$, its equation is, (Art. 23, Cor.)

$$y = \frac{y'}{x'}x\,;\ldots\ldots\ldots\ldots(1)$$

and since *BE* passes through $B(x''0)$ and is perpendicular to (1), its equation is (Art. 40, Cor 2)

$$y = -\frac{x'}{y'}(x - x'')\ldots\ldots\ldots\ldots(2)$$

Also, since *BC* passes through the points $B(x''0)$ and $C(x'y')$, its equation is

$$y = \frac{y'}{x' - x''}(x - x''),\ldots\ldots\ldots(3)$$

and since *AF* passes through the origin (0, 0) and is perpendicular to (3), its equation is

$$y = -\frac{x' - x''}{y'}x\ldots\ldots\ldots\ldots(4)$$

At the point where (2) and (4) intersect, their ordinates must be identical; hence, equating their values, we must have, at that point,

$$\frac{x'}{y'}(x - x'') = \frac{x' - x''}{y'}x,$$

whence, at the point of intersection,

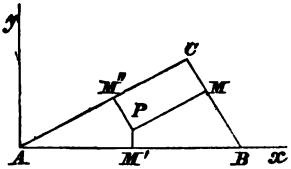
$$x = x'\,;$$

which proves the proposition.

The student may exercise himself by solving this problem with *DC* as axis of y instead of *Ay*: then, assuming *DC*, *DA*, *DB* to be known, he may express the equations to *AC* and *CB* in terms of the portions of the axes they cut off (Art. 19). It will then remain to prove that AF and BE intersect in a point whose abscissa $= 0$.

66. *To shew that the three perpendiculars through the middle points of the sides of a triangle meet in a point.*

Using the same axes and notation as in the last problem, we must find the co-ordinates of the three middle points M, M', M''; we can then find the equations to the two perpendiculars $MP, M''P$, from the condition that they each pass through a given point and are perpendicular to a given line; if we then shew that the abscissa of their point of intersection



$$= AM' = \frac{x''}{2},$$

we shall have proved that they intersect in the perpendicular from M'.

The co-ordinates of M'' are evidently $\frac{x'}{2}, \frac{y'}{2}$; also (Art. 36) the co-ordinates of M, the point of bisection of the line contained between the points $B\,(x''0)$ and $C\,(x'y')$, are

$$x = \frac{x' + x''}{2}, \quad y = \frac{y'}{2}.$$

As before, the equation to AC is

$$y = \frac{y'}{x'}x, \ldots\ldots\ldots\ldots\ldots(1)$$

and the equation to $M''P$ which passes through $M''\left(\frac{x'}{2}\frac{y'}{2}\right)$, and is perpendicular to (1), is

$$y - \frac{y'}{2} = -\frac{x'}{y'}\left(x - \frac{x'}{2}\right) \ldots\ldots\ldots(2)$$

Also, as before, the equation to BC is

$$y = \frac{y'}{x' - x''}(x - x''), \text{ ....}(3)$$

and the equation to MP which passes through $M\left(\frac{x' + x''}{2}\ \frac{y'}{2}\right)$ and is perpendicular to (3), is

$$y - \frac{y'}{2} = -\frac{x' - x''}{y'}\left(x - \frac{x' + x''}{2}\right) \text{......}(4)$$

At the point where (2) and (4) intersect, their ordinates must be identical; hence, equating their values, we have

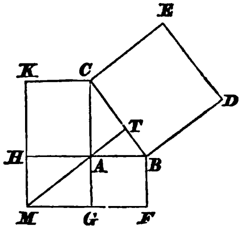
$$\frac{x'}{y'}\left(x - \frac{x'}{2}\right) = \frac{x' - x''}{y'}\left(x - \frac{x' + x''}{2}\right);$$

which gives $x = \frac{x''}{2}$,

as the abscissa of the point of intersection; but this abscissa belongs to some point in the perpendicular from M', which proves the proposition.

67. *In the figure of* Euc. I. 47, *if* KH *and* FG *be produced to meet in* M, *and* MA *produced to meet* BC *in* T, *shew that* MT *is perpendicular to* BC.

Take AB, AC produced indefinitely, as axes of x and y, and denote the sides of the triangle opposite to A, B, C by a, b, c; then the co-ordinates of M are $x = -b$, $y = -c$, and the equation to MA which passes through the origin and M is, (Art. 23, Cor.)



$$y = \frac{c}{b}x;$$

also the line BC cuts off from the axes of x and y intercepts $= c$ and b respectively; hence, (Art. 19) its equation is

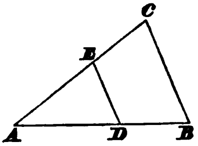
$$\frac{x}{c} + \frac{y}{b} = 1,$$

$$\text{or}\quad y = -\frac{b}{c}x + b,$$

which, by Art. 35, represents a line perpendicular to MA.

68. *To prove algebraically* Euc. VI. 2.

Let AB, AC the sides of the triangle be taken as oblique axes, and let $AB = c$, $AC = b$, $AE = d$, $AD = e$; then the equation to BC is (Art. 19), since it cuts off intercepts c and b on the axes of x and y respectively,



$$\frac{x}{c} + \frac{y}{b} = 1,$$

$$\text{or}\ y = -\frac{b}{c}x + b; \quad \ldots\ldots\ldots\ldots \quad (1)$$

and similarly the equation to DE is

$$\frac{x}{e} + \frac{y}{d} = 1,$$

$$\text{or}\ y = -\frac{d}{e}x + d; \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

hence, (Art. 37, Cor.), (2) is parallel to (1) if

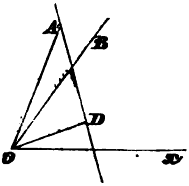
$$\frac{b}{c} = \frac{d}{e},$$

$$\text{or,}\ \frac{b-d}{d} = \frac{c-e}{e}:$$

i. e. if $EC : EA :: DB : AD$, and, conversely, if the lines (1) and (2) are parallel, these four lines are proportional.

69. *The sides containing a given angle are in a given ratio, and the vertex is fixed; supposing the extremity of one of the sides to move in a given straight line, to find the locus of the extremity of the other.*

Let OA, OB be the two sides; let angle $AOB = \alpha$, angle $AOx = \theta$, angle $BOx = \theta'$, Ox being any fixed straight line;



$$OA = \rho, \quad OB = \rho', \quad \rho' = n\rho.$$

Let A move in the straight line AD, the perpendicular on which (OD) from the origin $= p$ and makes an angle $DOx = \beta$ with Ox; then the polar equation to AD is (Art. 52),

$$\rho \cos(\theta - \beta) = p;$$

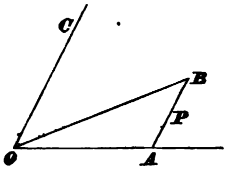
$$\text{hence } \rho' \cos(\theta - \beta) = np,$$

$$\text{or } \rho' \cos(\theta' - \overline{\beta - \alpha}) = np,$$

which is the polar equation to the locus of B, and is therefore a straight line, at a distance $= np$ from the origin, the inclination of this distance to Ox being $\beta - \alpha$.

70. *Given two fixed lines* OA *and* OB; *if any line* AB *be drawn to intersect these, parallel to a third fixed line* OC, *to find the locus of the point where* AB *is cut in a fixed ratio.*

Take OA, OC as axes of x and y, and let P be the point in AB, so that $AP = \frac{1}{n} AB$; then the equation to OB, since it passes through the origin, is (Art. 21, Cor. 3) of the form



$$y = mx,$$

and, since OA and AB are the abscissa and ordinate of the point B on the line,

$$AB = m \,.\, OA,$$

or, since $AB = n \,.\, AP$,

$$AP = \frac{m}{n} \,.\, OA,$$

and the equation to the locus of P is

$$y = \frac{m}{n} x,$$

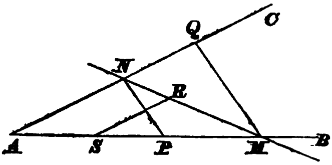
which represents a straight line through O.

71. *Given two straight lines* AB, AC *and a point* R, P *a point in* AB, *and* Q *a point in* AC; *it is required to draw through* R *a straight line which shall cut* AB *in* M *and* AC *in* N, *so that* PN *shall be parallel to* QM.

Take AB and AC as axes, and suppose the position of R to be determined by its co-ordinates $AS = h$, $RS = k$; let

$$AP = a, \quad AQ = b,$$

and assume $AM = x'$, $AN = y'$, which are quantities to be determined.



The equation to MN is

$$\frac{x}{x'} + \frac{y}{y'} = 1,$$

and since it passes through $R(hk)$, the co-ordinates of R must satisfy the equation; hence,

$$\frac{h}{x'} + \frac{k}{y'} = 1; \dots\dots\dots\dots (1)$$

also, since PN is parallel to QM,

$$AP : AM :: AN : AQ,$$

$$\text{or } \frac{a}{x'} = \frac{y'}{b} \text{............} (2)$$

From (1) and (2) x' and y', *i.e.* AM and AN may be determined.

EXAMPLES IV.

1. In the fig. of Euc. I. 5, if BG, CF meet in H, shew that AH bisects the angle BAC.

2. In the fig. of Euc. I. 47, shew that AL, BK, FC intersect in one point, and that, if BG and CH be joined, the lines will be parallel.

3. A line is drawn parallel to the base of a triangle, and its extremities joined transversely to the base; shew that the locus of the point of intersection of the joining lines is a line passing through the vertex and bisecting the base.

4. Let Ax bisect any straight line CD in O; draw CB, cutting Ax in B and AD produced in E; join DB and let it meet AC produced in F, and join FE; FE shall be parallel to DC.

5. In two given straight lines, drawn from a point O, take points P, Q in one, and P', Q' in the other, so that OP, OQ, OP', OQ' are in harmonical progression; shew that the locus of the intersections of PQ' and $P'Q$ is a line bisecting the angle between the given lines.

6. A side AB of a triangle, and a perpendicular through A, are assumed as the axes of x and y, and the equations to the other sides are $(BC) y = mx + b$, and $(AC) y = m'x$; determine the sides and angles of the triangle.

7. Taking the requisite data to fix a parallelogram in a plane, by equations to its sides, prove that the diagonals bisect each other.

8. $MANP$ is a parallelogram, having a given angle at A, and also its perimeter a given quantity; shew that the locus of P for all such parallelograms is a straight line.

9. The three bisectors of the sides of a triangle, drawn from the opposite angles, meet in a point.

10. The locus of a point, whose distances from two straight lines, of which the equations are given, are always to one another in a fixed ratio, is a straight line; axes being rectangular.

11. Given the base and the difference of the squares of the sides of a triangle; find the locus of the vertex.

12. Given the base and the sum of the sides of a triangle; if the perpendicular from the vertex on the base be produced through the vertex till its whole length equals one of the sides, shew that the locus of the extremity of the perpendicular is a straight line.

13. If Ax, Ay be two straight lines, and through any point P there be drawn straight lines PP_1Q_1, PP_2Q_2, &c. meeting Ax in P_1, P_2, &c., and Ay in Q_1, Q_2, &c., then, if AP_1, AP_2, &c. are in harmonical progression, so also are AQ_1, AQ_2, &c.

14. AB, AC are two straight lines given in position; a straight line DE meets them in D, E, respectively, so that $AD + AE$ is a constant length; also DE is divided in the point P so that DP bears a constant ratio to EP; the locus of P is a straight line.

SECTION VI.

The Circle.

72. Before examining the curves represented by the general equation of the second degree, we shall investigate some of the properties of the circle, which we shall afterwards see to be a particular case of a class of curves represented by that equation. We select the circle on account of its simplicity, and because the reader is already familiar with its principal properties, geometrically treated.

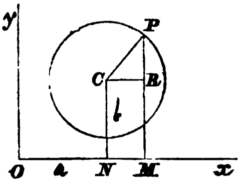
73. *To find the equation to a circle whose centre and radius are given, the co-ordinates being rectangular.*

If C (ab) be the centre of the circle, P any point (xy) on the circumference, and CR be drawn parallel to Ox to meet the ordinate of P in R, we have

$$CR^2 + PR^2 = CP^2,$$

$$\text{or } (x-a)^2 + (y-b)^2 = r^2,$$

where r = the radius of the circle.



This follows directly from Art. 6, and, in fact, the equation only asserts that the distance between the points (ab) and (xy) is constant and equal to r.

If the co-ordinates be oblique, and inclined to one another at an angle $= \omega$, we have, since angle CRP now $= 180 - \omega$,

$$(x-a)^2 + (y-b)^2 + 2(x-a).(y-b)\cos\omega = r^2;$$

but we shall seldom have occasion to use this equation.

74. Expanding the general equation to the circle referred to rectangular axes, we have

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0\,;$$

and hence it appears that the general equation to the circle is of the form

$$x^2 + y^2 + Ax + By + C = 0,$$

A, B, C being any constants. The equation

$$Ax^2 + Ay^2 + Bx + Cy + D = 0,$$

may be reduced to this form by dividing by A, and is therefore the most general form that the equation can assume, when the co-ordinates are rectangular.

75. If we can reduce an equation to the form

$$x^2 + y^2 + Ax + By + C = 0,$$

we may determine the circle it represents; for, adding $\frac{A^2}{4} + \frac{B^2}{4}$ to both sides of the equation, we have

$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 = \frac{A^2}{4} + \frac{B^2}{4} - C,$$

which is evidently the equation to a circle, the co-ordinates of whose centre are

$x = -\frac{A}{2}$, $y = -\frac{B}{2}$, and whose radius $= \left(\frac{A^2}{4} + \frac{B^2}{4} - C\right)$.

If the quantity $\frac{A^2}{4} + \frac{B^2}{4} - C$ be $= 0$, the equation may be considered to represent a circle with an infinitely small radius, or, in fact, the centre, for the equation is now

$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 = 0,$$

which can only be satisfied by the values $x = -\frac{A}{2}$, $y = -\frac{B}{2}$;

if $\frac{A^2}{4} + \frac{B^2}{4} - C$ be negative, the radius is an impossible quantity, and the circle imaginary.

Ex. 1. The equation $x^2 + y^2 - 2x + 4y + 1 = 0$, may be written $(x - 1)^2 + (y + 2)^2 = 4$, which represents a circle the co-ordinates of whose centre are $x = 1$, $y = -2$, and whose radius $= 2$.

Ex. 2. The equation $x^2 + y^2 + 2x - 6y + 10 = 0$, may be written $(x + 1)^2 + (y - 3)^2 = 0$, a circle the co-ordinates of whose centre are $x = -1$, $y = 3$, and whose radius $= 0$.

Ex. 3. $x^2 + y^2 + 2x + 6y + 11 = 0$, or $(x + 1)^2 + (y + 3)^2 = -1$, represents an imaginary circle.

76. If in the equation $(x - a)^2 + (y - b)^2 = r^2$, $a = 0$, $b = 0$, or the centre of the circle be origin, the equation becomes

$$x^2 + y^2 = r^2.$$

If $a = r$, $b = 0$, or a diameter be chosen as axis of x, and its extremity as origin, the equation becomes

$$x^2 - 2rx + y^2 = 0,$$

and similarly, if the axis of y be a diameter and the origin at its extremity, the equation is

$$x^2 + y^2 - 2ry = 0.$$

77. It may be observed here, that, in the circle, as well as every other curve, if the origin is on the curve, there will be no term which does not involve either x or y; for the equation must be satisfied by the values $x = 0$, $y = 0$, which cannot be the case if there be a term which does not vanish when x and y vanish.

78. If we expand the equation to the circle (Art. 73), referred to oblique axes, we obtain

$$x^2 + y^2 + 2\cos\omega . xy - 2(a + b\cos\omega)\,x - 2(b + a\cos\omega)\,y$$
$$+ a^2 + b^2 + 2ab\cos\omega - r^2 = 0 \;\ldots\ldots\; (1).$$

Hence, if the inclination of the axes be ω, the general equation to the circle is

$$x^2 + y^2 + 2\cos\omega . xy + Ax + By + C = 0 \;\ldots\ldots\ldots\; (2),$$

where A, B, C are constants.

Ex. *To determine the inclination of the co-ordinate axes, in order that the equation*

$$x^2 - xy + y^2 - ax - ay = 0$$

may represent a circle, and find the magnitude of its radius.

Comparing the equation with equation (1), we have the equations

$$2\cos\omega = -1,\;\; a^2 + b^2 + 2ab\cos\omega - r^2 = 0,$$
$$2(a + b\cos\omega) = a = 2(b + a\cos\omega),$$

from which we obtain

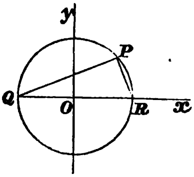
$$\omega = \frac{2}{3}\pi,\;\; a = b = r = a.$$

79. The equation

$$x^2 + y^2 = r^2$$

will give us a well-known property of the circle; for it may be obtained by eliminating k, by means of multiplication, from the equations

$$y = k(x - r), \;\ldots\ldots\ldots\; (1),$$
$$y = -\frac{1}{k}(x + r), \;\ldots\ldots\; (2),$$



where k is a constant, and perfectly arbitrary.

But these equations evidently represent lines which

(i) pass through the extremities of the diameter QR which is the axis of x; for (Art. 35), equation (1) represents a line passing through the intersection of the lines $y = 0$, $x - r = 0$, which is the point R, and (2) passes through the intersection of $y = 0$, $x + r = 0$, which is the point Q.

(ii) intersect in the circle, since, by eliminating k between them, we have the equation to the circle; and

(iii) are at right angles to one another by Art. 37; and they represent all lines which fulfil these three conditions. Hence we see that the locus of the vertices of all right-angled triangles on QR as base is the semicircle QPR.

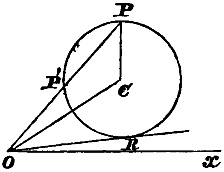
80. *To find the equation to the circle referred to polar co-ordinates.*

Let Ox be the initial line, O the pole; let the co-ordinates of the centre C be the known quantities ρ', θ', and of any point P, in the circumference, ρ, θ; then

$$CO^2 + PO^2 - 2\,CO.PO.\cos POC = CP^2,$$

$$\text{or } \rho'^2 + \rho^2 - 2\rho'\rho\cos(\theta - \theta') = r^2,$$

which is the polar equation required.



It will be seen that this is the formula of Art. 54, and only asserts that the distance between the points, whose polar co-ordinates are ρ, θ, and ρ', θ', is constant and equal to r.

Cor. 1. The two values of ρ which may be found from the equation

$$\rho^2 - 2\rho'\cos(\theta - \theta').\rho + \rho'^2 - r^2 = 0.$$

are the two distances from the pole O of the points P, P', where the radius vector, which makes an angle θ with Ox, cuts the circle. The product of the roots of this equation (Appendix) $= \rho'^2 - r^2$, a quantity which does not change for different values of θ. Hence the product rectangle $OP \,.\, OP'$ is constant for all positions of OP. When the roots are equal, or the line touches the circle, as OR, we have

$$\rho'^2 - r^2 = OR^2.$$

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Hence $OP \, . \, OP' = OR^2$, as in ~~Exercises 35, 36~~.

Cor. 2. If $\rho' = 0$, or the centre be pole, the equation becomes

$$\rho = r.$$

Cor. 3. If $\rho' = r$ and $\theta' = 0$, or a diameter be the initial line, and one extremity of it the pole, the equation becomes

$$\rho = 2r \cos \theta.$$

The reader will do well to verify by geometrical figures the results obtained here and in Art. 76.

81. In the foregoing Articles we have assumed one only of the well-known geometrical properties of the circle, viz. that the distance from the centre to the circumference is constant, and from this property we have deduced the equation. Most of the following Articles will admit of being proved in a very simple manner by those properties of the circle with which the reader is familiar; but we prefer to deduce our proofs from the equation alone, because this method is the same as that which we shall use in the case of other curves, and it is desirable that the student should perceive that all the properties of the circle may be obtained from its equation, without any previous acquaintance with the curve. It will, however, be an exercise very profitable to the student, if he endeavour to deduce the equations of the following Articles from any of the properties of tangents, &c. which he may find in Euclid.

tangent to

82. *To find the equation to a line touching the circle at a point* (x′y′).

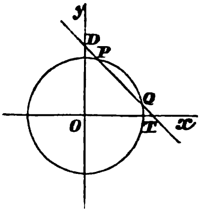
Let DT be a line cutting the circle

$$x^2 + y^2 = r^2 \quad . \quad . \quad . \quad . \quad (1)$$

in the point P $(x'y')$; then the equation to DT is (Art. 27)

$$\frac{y - y'}{s} = \frac{x - x'}{c} = l. \quad . \quad . \quad . \quad (2)$$

No 82 is very important



Now, if we substitute for x and y from (2) in (1), the result will be a quadratic in l, the two roots of which equation will be the distances of the point $(x'y')$ from the points where DT cuts the circle. Hence from (1) and (2) we have

$$(cl + x')^2 + (sl + y')^2 = r^2,$$

$$\text{or } (c^2 + s^2)\, l^2 + 2\,(cx' + sy')l = 0, \quad . \quad . \quad . \quad (3)$$

$$\text{since } x'^2 + y'^2 = r^2,$$

$(x'y')$ being a point on the circle.

Equation (3) gives us

$$l = 0,$$

(as it should, for $(x'y')$ coincides with one of the points of section,) and also

$$(c^2 + s^2)\, l + 2\,(cx' + sy') = 0,$$

the value of l in which is the distance PQ. But, if we suppose the point Q to move up to P, this distance vanishes, and the line becomes a tangent at $P(x'y')$, and we have, as the condition that (2) should be tangent,

as far as here applies to a secant

$$cx' + sy' = 0. \quad . \quad . \quad . \quad . \quad (4)$$

Eliminating c and s by means of this equation and the equation to the line, we have

$$(x - x')x' + (y - y')y' = 0, \quad . \quad . \quad . \quad (5)$$

$$\text{or } xx' + yy' - (x'^2 + y'^2) = 0,$$

whence $$xx' + yy' = r^2,$$

which is the equation to the tangent at the point $(x'y')$.

We leave the reader to obtain this equation geometrically, by means of known properties of the tangent. For

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example, he will find it easy to shew, from Euc. III. 16, Cor. that

$$\frac{x - x'}{y - y'} = -\frac{y'}{x'},$$

which gives us equation (5) of this Article.

83. If we transfer the origin to any point $(-a, -b)$ so that the co-ordinates of the centre are a and b, we must write in the equation to the tangent, by Art. 59,

$$x' - a,\ x - a,\ y' - b,\ y - b \text{ for } x',\ x,\ y',\ y,$$

respectively, and the equation becomes

$$(x - a)(x' - a) + (y - b)(y' - b) = r^2.$$

84. If the problem be to find the equation to a tangent which makes a *given angle* with the axis of x, we must eliminate x' and y', instead of c and s, from equation (2).

We have from (4),

$$\frac{s}{c} = -\frac{x'}{y'};$$

$$\text{whence } \frac{s^2 + c^2}{c^2} = \frac{x'^2 + y'^2}{y'^2},$$

$$\frac{1}{c^2} = \frac{r^2}{y'^2},$$

$cx' + sy' = 0$

$cx' \pm src = 0$

$x \pm rs = 0$

$x = \mp rs$

from which and from (4), we have

$$y' = \pm rc, \qquad x' = \mp rs;$$

hence (2) becomes

$$\begin{aligned} cy - sx &= (cy' - sx') \\ &= \pm r(c^2 + s^2) \\ &= \pm r; \end{aligned}$$

$y = \frac{sx}{c} \pm \frac{r}{c}$

$y = mx + \frac{r}{c}$

or, if $\frac{s}{c} = m$, where m is the trigonometrical tangent of the angle which the tangent makes with the axis,

$$y = mx \pm r\sqrt{1 + m^2};$$

$c^2 + s^2 = 1$

$\frac{1}{c^2} = \frac{c^2}{c^2} + \frac{s^2}{c^2} = 1 + m^2$

$\frac{1}{c} = \sqrt{1 + m^2}$

but we shall obtain this equation directly, without introducing the point $(x'y')$, in Article 86.

85. The equation to the tangent in terms of its distance from the centre and the angle (a) which that distance makes with the axis of x is evidently

$$x \cos a + y \sin a = r,$$

since the perpendicular from the centre is the radius. This may be deduced from the equation of Art. 84, without assuming the above property of the circle.

86. *To find the equation to the tangent in terms of its inclination to the axis of* x.

Let $y = mx + b$, $x^2 + y^2 = r^2$, represent a straight line and a circle respectively; if we find values of x and y which satisfy both these equations, these will evidently be the co-ordinates of a point where the line and circle intersect, for no co-ordinates but those of the points of section can be common to the line and circle. Eliminating x between them, we obtain

$$y^2(1 + m^2) - 2by + b^2 - m^2r^2 = 0,$$

the roots of which equation are the ordinates of the points of intersection. If these roots be equal, or the line touch the curve instead of cutting it, we have (Appendix)

$$b^2 = (b^2 - m^2r^2).(1 + m^2),$$

$$\text{or } b = \pm r\sqrt{1 + m^2},$$

and the equation to the tangent which makes with the axis of x an angle $= \tan^{-1}m$, is

$$y = mx \pm r\sqrt{1 + m^2},$$

$m = \tan \alpha$ $\tan^{-1}$ = angle of which the tang. is.

$= \tan^{-1} m$.

the double sign referring to the two tangents at the extremities of any diameter, which are parallel.

Cor. 1. Hence the condition that any line $y = mx + b$ should touch the circle $x^2 + y^2 = r^2$, is

$$b^2 = r^2(1 + m^2).$$

87. *To determine the equations to the tangents drawn to a circle from any point* x′y′ *without it.*

Let the equation to the tangent be

$$y - mx = \pm r\sqrt{1 + m^2};$$

then, since it passes through $(x'y')$, the co-ordinates of that point satisfy the equation, and we have

$$(y' - mx')^2 = r^2(1 + m^2),$$

$$\text{or } (x'^2 - r^2)m^2 - 2x'y'm + y'^2 - r^2 = 0, \quad . \quad . \quad . \quad (1)$$

a quadratic to determine the two values of m in the equations to the two tangents which can be drawn to the circle from $(x'y')$. If m and m' be the two roots, the required equations will be (Art. 22, Cor. 1)

$$y - y' = m(x - x'), \quad y - y' = m'(x - x').$$

Cor. Solving equation (1), we have

$$m = \frac{x'y' \pm r\sqrt{x'^2 + y'^2 - r^2}}{x'^2 - r^2},$$

whence we see that the tangents are real if $x'^2 + y'^2 > r^2$, or the point is outside the circle; they coincide when $x'^2 + y'^2 = r^2$, or the point is on the circle; and they are imaginary when $x'^2 + y'^2 < r^2$, or the point is within the circle.

88. The line drawn through any point in a curve perpendicular to the tangent is called the *Normal.*

In the case of the circle, if the point be $(x'y')$, the equation to the normal is

Section 89 to 93 Important.

$$y - y' = m(x - x'),$$

where m must (Art. 37) have a value, the reciprocal of the value found in the case of the tangent, and must be of a different sign; hence, (Art. 84) $m = \frac{y'}{x'}$, and the equation is

$$y - y' = \frac{y'}{x'}(x - x'),$$

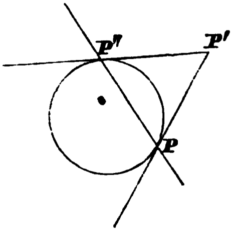
which, after reduction, becomes

$$y = \frac{y'}{x'}x,$$

the equation to a straight line passing through the origin; hence every normal in the circle passes through the centre, as is proved in Euc. III. 16, 19. Serm. Sec. 120

89. *To find the equation to the chord joining the points of contact of two tangents from any point* (x′y′).

Let $P'(x'y')$ be the external point, and let PP'' be the line whose equation is required.



Now the equation to $P'P''$, the tangent at $P''(x''y'')$, is (Art. 83)

$$xx'' + yy'' = r^2,$$

and, since this line passes through $P'(x'y')$, the co-ordinates of P' satisfy the equation, hence,

$$x'x'' + y'y'' = r^2;$$

hence the values $x = x''$, $y = y''$ satisfy the equation

$$xx' + yy' = r^2,$$

or $P''(x''y'')$ is a point on the line it represents; and by exactly similar reasoning, P, the other point of contact, is on this same straight line; hence

$$xx' + yy' = r^2$$

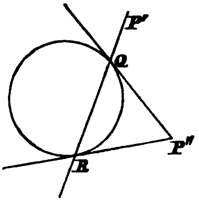
is the equation to the chord joining the points of contact of tangents drawn from the point $(x'y')$; for it is the equation to *some* straight line, and both P and P'' have been proved to lie in it.

Cor. Hence, to draw tangents to the circle from any external point $(x'y')$, we have the two equations

$$xx' + yy' = r^2, \quad x^2 + y^2 = r^2,$$

to determine the co-ordinates of the points of contact. These equations will always give two points, corresponding to the points of intersection of the line and circle.

90. *A chord of a circle is drawn through a fixed point* $(x'y')$, *and tangents are drawn at the points where it cuts the circle; to find the equation to the locus of the intersection of these tangents when the chord is turned about the point* $(x'y')$.



Let P' be the point $(x'y')$, $P'QR$ the chord, and let the tangents at Q and R intersect in $P''(x''y'')$; it is required to find the locus of P'' as the chord turns about P'. Considering $P'R$ as the chord joining the points of contact of tangents drawn from the point $(x''y'')$, its equation is (Art. 89)

$$xx'' + yy'' = r^2, \quad . \quad . \quad . \quad (1)$$

but, since it passes through $P'(x'y')$ we have

$$x'x'' + y'y'' = r^2; \quad . \quad . \quad . \quad . \quad (2)$$

but $(x''y'')$ is *any* point in the required locus, and its co-ordinates satisfy the equation

$$xx' + yy' = r^2, \quad . \quad . \quad . \quad . \quad (3)$$

hence, the co-ordinates of every point in the locus satisfy this equation, which is, therefore, the equation required.

The locus is therefore a straight line, evidently perpendicular to the line $y = \frac{y'}{x'}x$, *i. e.* to the line joining the centre with the point $(x'y')$.

This line is called the *polar* of the point $(x'y')$ with regard to the circle $(x^2 + y^2 = r^2)$, and the point $(x'y')$ is called the *pole* of the line. These terms must not in any way be confounded with the same terms used in Arts. 49—54.

91. The reasoning of the preceding Article is very often perplexing to the beginner. The difficulty commonly arises from the use of the co-ordinates of P'' $(x''y'')$ as *constants* in equation (1), and afterwards as *variables* in equation (3). We should bear in mind that, although P'' is a moveable point, we do not examine its position during its motion, but, taking it in any *one* of its several positions, we obtain a relation between its co-ordinates *while in that position*. The relation so obtained is equally true for all positions, and is therefore the equation to the locus of the point. Thus, equations (1) and (2) are relations obtained between the coordinates of the actual points P', P''. Equation (3) declares this relation to be true for *all* points determined by the same law as P''.

92. As the equation to the polar is one of the greatest importance, and will be frequently used in the following pages, we will define its exact meaning in all cases.

(1) The position of the point P' (Art. 90) is not subject to any limitation; hence, wherever the point $(x'y')$ may be, the equation $xx' + yy' = r^2$ represents the locus of the intersection of tangents drawn at the extremities of chords which all pass through $(x'y')$.

(2) If the point be *without* the circle, this locus is

(Art. 89) identical with the chord joining the points of contact of tangents drawn from $(x'y')$.

(3) If the point be *on* the circle, the locus is also (Art. 82) the tangent at the point $(x'y')$.

93. *If one point* (x′y′) *lie on the polar of a second point* (x″y″), *then will the point* (x″y″) *lie on the polar of the point* (x′y′).

For the equation to the polar of $(x''y'')$ is

$$xx'' + yy'' = r^2,$$

and the condition that $(x'y')$ should lie on this line, is

$$x'x'' + y'y'' = r^2;$$

but this is also the condition to be fulfilled in order that the point $(x''y'')$ should lie on the line

$$xx' + yy' = r^2.$$

94. *To find the locus of the middle points of any system of parallel chords in a circle.*

$$\text{Let} \quad \frac{x - x'}{c} = \frac{y - y'}{s} = l$$

be the equation to any one of the chords of the system, $(x'y')$ its middle point; then, for its intersection with the circle, $(x^2 + y^2 = r^2)$, we have

$$(cl + x')^2 + (sl + y')^2 = r^2,$$

$$\text{or} \quad (c^2 + s^2)\, l^2 + 2\,(cx' + sy')\, l + x'^2 + y'^2 = r^2,$$

l being the distance from the middle point $(x'y')$ to the points of section of line and circle; hence, (Appendix), the roots of this equation are equal and of opposite signs,

$$\therefore\; cx' + sy' = 0,$$

a relation which is true for the co-ordinates of every middle point, since c and s are the same for every chord of the system, and therefore the equation

$$cx + sy = 0$$

is the equation to the locus of the middle points, and evidently represents a straight line through the centre and perpendicular to the chords.

95. In working problems on the circle, it is sometimes convenient to express the position of any point $(x'y')$ on the circumference by means of a single variable. If θ' be the angle which the radius vector of the point makes with the axis of x, we shall have

$$x' = r\cos\theta', \quad y' = r\sin\theta'.$$

Ex. *To find the co-ordinates of the point of intersection of two tangents to the circle at two given points.*

The equations to the tangents will be

$$x\cos\theta' + y\sin\theta' = r, \quad x\cos\theta'' + y\sin\theta'' = r,$$

and the co-ordinates of their intersection are

x. 37. 44. 46. 53
in Trigonometry.

$$x = r\frac{\cos\frac{1}{2}(\theta' + \theta'')}{\cos\frac{1}{2}(\theta' - \theta'')}, \quad y = r\frac{\sin\frac{1}{2}(\theta' + \theta'')}{\cos\frac{1}{2}(\theta' - \theta'')}.$$

96. *The equations to two circles are given; to find the equation of the straight line which joins their points of intersection.*

Let the equations to the two circles be

$$x^2 + y^2 + Ax + By + C = 0, \quad . \quad . \quad . \quad (1)$$

$$x^2 + y^2 + A'x + B'y + C' = 0; \quad . \quad . \quad . \quad (2)$$

then, by Art. 35, if k be any constant,

$$x^2 + y^2 + Ax + By + C + k(x^2 + y^2 + A'x + B'y + C') = 0 \quad . \quad (3)$$

is the equation to some curve passing through the points where (1) and (2) intersect, since it is satisfied by those values of x and y which satisfy (1) and (2). The equation may be written

$$(1+k)x^2+(1+k)y^2+(A+kA')x+(B+kB')y+C+kC'=0,$$

which evidently represents a circle (Art. 75) since it can be reduced to the form

$$x^2+y^2+Px+Qy+R=0,$$

by dividing by $1+k$.

If $k=-1$, or (2) be simply subtracted from (1), the equation becomes

$$(A-A')x+(B-B')y+C-C'=0,$$

which is therefore the equation to a *straight line* passing through the intersections of (1) and (2).

97. *To find the length of a line drawn from a point* (x′y′) *to meet a circle, axes being rectangular.*

Let the equation to the circle be

$$(x-a)^2+(y-b)^2-r^2=0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We shall, for the sake of brevity, write this

$$S=0.$$

Let the equation to the line be

$$\frac{y-y'}{s}=\frac{x-x'}{c}=l. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting for x and y from (2) in (1), and remembering that $s^2+c^2=1$, we have

$$l^2+2\{c(x'-a)+s(y'-b)\}l+(x'-a)^2+(y'-b)^2-r^2=0. \quad . \quad (3)$$

If the roots of this equation be not equal, or the line cut the circle, the rectangle contained by the segments of the line (Appendix)

$$= (x' - a)^2 + (y' - b)^2 - r^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and, if the roots be equal, or the line touch the circle, this expression is equal to the square of the tangent. Hence, if $S = 0$ be the equation to a circle, the length of the tangent from an exterior point (xy) is equal to S. It may easily be seen that expression (4) becomes negative when the point $(x'y')$ is within the circle, and that the roots cannot in that case be equal.

98. We have seen, by Art. 96, that, if S and S' are symbols standing for the expressions

$$(x - a)^2 + (y - b)^2 - r^2, \quad (x - a')^2 + (y - b')^2 - r'^2,$$

so that

$$S = 0, \ldots\ldots (1) \qquad S' = 0 \ldots\ldots (2)$$

are the equations to two circles, then the equation

$$S - S' = 0 \ldots\ldots (3)$$

is the equation to a straight line, and that, if the circles intersect, the points of intersection will lie on (3). But (3) is a real line, whether the circles intersect or not, and in either case possesses the following important property with regard to the circles; *if from any point of it, lines be drawn to touch both circles, the lengths of those lines are equal;* for the lengths of the tangents to (1) and (2) from any point (xy) on (3) are (Art. 97) equal to S and S' respectively, and (3) asserts that these are equal.

If (1) and (2) be in their simplest forms, *i. e.* if the coefficients of x^2 and y^2 in them are unity, the line (3) is called the *Radical Axis* of the two circles.

99. If there be two unequal circles which do not intersect, there will evidently be two points, on the indefinite

straight line joining their centres, from each of which a pair of tangents common to the two circles may be drawn. The points will lie, one between the centres, and the other exterior to the smaller circle. They are called the *Centres of Similitude*[*].

EXAMPLES V.

1. To find the centre and radius of the circle

$$x^2 + y^2 - 6x + 4y + 4 = 0.$$

2. Investigate the line or lines represented by the equation

$$x^3 + xy^2 - x^2r - xr^2 - ry^2 + r^3 = 0.$$

3. Find the common chord of two circles

$$(x-1)^2 + (y-2)^2 = 6, \quad (x-2)^2 + (y-3)^2 = 8.$$

4. To find the equation to a straight line which passes through the centres of the two circles

$$x^2 + 2x + y^2 = 0, \quad y^2 + 2y + x^2 = 0.$$

5. To find the equation to a circle having for its diameter the line joining the points of intersection of the line, $y = ax$, and the circle, $y^2 = 2rx - x^2$.

6. Find the equation to the circle, the diameter of which is the common chord of the circles

$$x^2 + y^2 = r^2, \quad (x-a)^2 + y^2 = r^2.$$

7. What is represented by the equation

$$x(x-2) + y(y-4) + 8 = 0\,?$$

8. Find a relation between the coefficients of the equation

$$A(x^2 + y^2) + Dx + Ey + F = 0,$$

in order that (1) the axis of x, and (2) the axis of y, may be tangents to the circle.

[*] The properties of these points are discussed at length in Salmon's *Conic Sections*.

9. To find the inclination to the axis of x of the tangents drawn from any point $(x'y')$ to the circle whose equation is

$$(x-a)^2+(y-b)^2-r^2=0.$$

10. To find the relation between the quantities a, b, r, in order that the line $\frac{x}{a}+\frac{y}{b}=1$ may touch the circle $x^2+y^2=r^2$.

11. To find the equation to a circle, the centre of which is at the origin of co-ordinates, and which is touched by the line $y=2x+3$.

12. To find the intercepts on the axes of co-ordinates of the tangent to a circle $(x^2+y^2=r^2)$, drawn parallel to a given straight line,

$$(x\cos\alpha+y\sin\alpha=p).$$

13. If $2\alpha'$, $2\alpha''$ be the inclination of two radii of a circle, $x^2+y^2=r^2$, to the axis of x, to find the equation to the chord joining the extremities of the radii.

14. If a chord of constant length be inscribed in a circle, it will always touch another circle.

15. Find the locus of a point where a chord of constant length is cut in a constant ratio.

16. To find the locus of the middle points of chords of a given circle drawn parallel to a given line $(y=mx+b)$.

17. To find the locus of the centre of the inscribed circle, when the base and vertical angle of a triangle are given.

18. Find the locus of the middle points of chords drawn from the extremity of the diameter of any circle.

19. If circles be described on the sides of a triangle as diameters, they will intersect in the points where the perpendiculars from the opposite angles meet the sides, or the sides produced.

20. The three radical axes belonging to three given circles, taken two and two, pass through one point.

21. If a chord be drawn parallel to the diameter of a circle, and, from any point in the diameter, lines be drawn to its extremities, the

sum of their squares will be equal to the sum of the squares of the segments of the diameter.

22. Find the locus of the vertices of all triangles which have a given base, and a given vertical angle.

23. Prove Euc. III. 31, from the resulting equation.

24. The vertical angle of any oblique-angled triangle inscribed in a circle is greater or less than a right angle by the angle contained by the base and the diameter drawn from the extremity of the base.

25. Tangents are drawn to a circle $x^2 + y^2 = r^2$, at two points $(x'y')$, $(x''y'')$; to find the distance of a point (hk) from a line passing through the centre and the intersection of the two tangents.

26. To find the equations to straight lines touching the circle

$$x^2 + y^2 = 10,$$

at points, the common abscissa of which is unity.

27. Find the equation to a line touching a given circle

$$(x - a)^2 + (y - b)^2 = r^2,$$

and parallel to a given line $y = mx + c$.

28. Find the equation to a circle having for diameter the distance of a given point $(x'y')$ from the origin of co-ordinates.

29. To find the equation to the straight line passing through the origin of co-ordinates, and touching the circle $x^2 + y^2 - 3x + 4y = 0$.

30. To find the length of the common chord of the circles

$$(x - \alpha)^2 + (y - \beta)^2 = \gamma^2, \quad (x - \beta)^2 + (y - \alpha)^2 = \gamma^2.$$

31. To prove that all circles represented by the equation

$$(x - \alpha)^2 + (y - \beta)^2 - \gamma^2 + k\{(x - \alpha_1)^2 + (y - \beta_1)^2 - \gamma_1^2\} = 0,$$

where α, β, γ, α_1, β_1, γ_1 are constants, and k any quantity whatever, pass through two fixed points, and to find the equation to the straight line passing through the points.

32. If a circle be described on the radius of another circle, any chord of the outer circle, drawn from the point of contact, will be bisected by the inner.

33. To find the length of the chord of a circle $x^2 + y^2 = r^2$ made by the straight line $\frac{x}{a} + \frac{y}{b} = 1$.

34. Through any fixed point A in the circumference of a circle draw chords AP, AQ at right angles, and join PQ. If O be a point in PQ such that $PO = n \,.\, PQ$, the locus of O is a circle.

35. Shew that, by giving a suitable value to k, the equation $S - kS'$ may be made to represent *any* circle passing through the intersections of $S = 0$, $S' = 0$.

36. Given a line and a circle, to find a point, such that, if any chord be drawn through it, and perpendiculars let fall from its extremities on the given line, the rectangle under these perpendiculars will be constant.

37. To determine the relations between the constants of the two equations $lx + my = \delta$, and $(x - a)^2 + (y - b)^2 = r^2$ in order that they may touch each other.

38. If from a given point S, a perpendicular be drawn to the tangent PY at any point P of a circle, of which the centre is C, and, in the line MP at right angles to CS and produced if necessary, a point Q be taken such that $QM = SY$, to find the locus of Q.

39. Given the equation to a circle, and the chord of a circle; shew that a perpendicular let fall upon the chord from the centre bisects the chord.

40. Find the diameter of the circle

$$x^2 + y^2 + 2xy \cos \omega = ax + by.$$

41. The radical axis is perpendicular to the line joining the centres of two circles.

42. Find the equation to the circle which passes through the points $(0, 0)$, $(-8a, 0)$, $(0, 6a)$.

43. To find the locus of middle points of chords which pass through a given point.

44. If on any radius vector through a fixed point O, OQ be taken in a constant ratio to OP, find the locus of Q.

45. The circles represented by the equation

$$(n+1).(x^2+y^2)=ax+nby,$$

where n is arbitrary, have a common chord.

46. To prove analytically that the angles in the same segment of a circle are equal.

47. Prove analytically that the angle in a semicircle is a right angle.

48. Given the base and vertical angle, to shew that the locus of the point of intersection of the perpendiculars of the triangle is a circle.

49. Given base and ratio of sides of a triangle; shew that the locus of the vertex is a circle.

50. When will the locus of a point be a circle, if the square of its distance from the base of a triangle be in a constant ratio to the product of its distances from the sides?

51. When will the locus of a point be a circle, if the sum of the squares of the three perpendiculars from it on the sides of any triangle be constant?

52. Find the locus of a point, the square of whose distance from a given point is proportional to its distance from a given right line.

53. Given the base of a triangle, and m times the square of one of its sides $\pm n$ times the square of the other $=$ a constant; find locus of the vertex, find centre and radius of resulting circle, and where it cuts base.

54. Find the equations to the circles which touch the three lines

$$x=a, \quad y=2b, \quad y=2b'.$$

55. To find the equation to a circle which passes through three points, the co-ordinates of which are (1, 2), (1, 3), (2, 5).

56. Trace the circle whose equation is

$$a(x^2+y^2)+b(x+y)=0.$$

57. The locus of the centres of all circles inscribed in all right-angled triangles on the same hypotenuse is the quadrant described on the hypotenuse.

58. The equation to a circle is $y^2+x^2=a(y+x)$; what is the equation to that diameter which passes through the origin of co-ordinates?

59. To find the equation to a circle referred to two tangents at right angles, as axes.

60. If through any point of a quadrant whose radius is R, two circles be drawn, touching the bounding radii of the quadrant, and r, r' be the radii of these circles, $rr'=R^2$.

61. To find the equations to the lines which touch both the circles

$$x^2+y^2=r^2,\quad (x-a)^2+y^2=r'^2.$$

62. To find the equation to the circle which touches the three lines

$$x=0,\quad y=0,\quad \frac{x}{a}+\frac{y}{b}=1.$$

63. To find the equations to two circles, which touch rectangular axes of x and y, and pass through a given point (ab).

64. APB is the segment of a circle. Take any point P and let PC be drawn bisecting the angle APB; shew that the dividing lines all pass through one point in the circumference of the circle of which APB is a segment.

65. The lines joining the angles of a triangle with the points in which the escribed circles touch the opposite sides, meet in a point.

66. A rod of given length is moved between two fixed lines, CP and CQ; the perpendiculars upon CP, CQ from P, Q meet in R, and those from P, Q on CQ, CP meet in S; shew that the loci of R and S are circles with common centre C.

67. In any circle draw a chord AB; from the middle point E of the lesser segment draw any line cutting AB in C and meeting the circumference in D; join AD, and in AD take $AP=AC$; find locus of P.

68. The axes Ox, Oy cut a circle in points A, A', B, B' respectively; to compare the values of x, y, at the intersection of the chords AB', $A'B$.

69. Determine the magnitude and position of the circle

$$\rho^2 - 2\rho(\cos\theta + \sqrt{3}\sin\theta) = 5.$$

70. Find the locus of a point such that, if straight lines be drawn to it from the four corners of a given square, the sum of their squares may be invariable.

71. ABC is an equilateral Δ, and

$$PA = PB + PC;$$

shew that P lies on the circle which circumscribes the Δ.

72. ACB is the segment of a circle, and any chord AC is produced to a point P, so that $AC : CP$ in a given ratio; required to find the locus of P.

73. Find the equation to a straight line which cuts a given circle, when the lines drawn from the points of intersection to the centre contain a right angle, and one of them is inclined to the axis of x at a given angle.

74. Two straight lines revolve uniformly in one plane about one extremity, the one moving twice as fast as the other. Find the locus of their points of intersection, supposing them to begin to move together in the same direction, from the straight line joining their fixed extremities.

75. If any number of circles touch one another in one point, all their polars, which correspond to a common pole, pass through a single point.

76. To find the locus of the pole, when the polar of a given circle always passes through a given point.

77. In the sides AB, AC of a given triangle ABC, take two points M, N, such that $\dfrac{MB}{AM} = \dfrac{AN}{NC}$, and shew that the circle described about the triangle AMN will always pass through a given point.

SECTION VII.

General Equation of the Second Degree.

100. THE equation of the first degree has given us but one species of line, viz. the straight line. We saw, in the case of the circle, that an equation of the second degree may represent a curve limited in every direction, and the case where it represents two straight lines, shews us that it may represent loci extending to infinity. These are but particular cases of the equation of this degree, of which the most general form is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

We shewed (Art. 17), by dividing the equation by the coefficient of one of the variables, that there are never more than two constants really involved in the equation of the first degree, and in like manner we may see that there are never more than five constants involved in the equation of the second degree, for, dividing by F, the equation becomes

$$\frac{A}{F}x^2 + \frac{B}{F}xy + \frac{C}{F}y^2 + \frac{D}{F}x + \frac{E}{F}y + 1 = 0;$$

which has only five constants $\frac{A}{F}$, $\frac{B}{F}$, &c. Hence, a curve of the second order may be made to fulfil *five* conditions; for instance, it may be described through five points, for the co-ordinates of those points substituted in the equation successively will give five equations to determine $\frac{A}{F}$, $\frac{B}{F}$, &c.

101. Solving the general equation so as to obtain y in terms of x, we have

$$y = \frac{-(Bx+E) \pm \sqrt{(B^2-4AC)x^2+2(BE-2CD)x+E^2-4CF}}{2C}.$$

By reason of the double sign of the radical, there are, in general, two values of y for every value of x, *i. e.* there are two ordinates which correspond to the same abscissa. As long as the values given to x render the quantity under the root positive, we can calculate and construct these ordinates, and so determine two points in the curve corresponding to each abscissa; if they make it $= 0$, the two points are coincident; and lastly, if they make it negative, the value of y is imaginary, or no point of the curve corresponds to the abscissa in question. Hence, to find the extent and limits of the curve in a direction parallel to the axis of x we must examine what values of x render the radical part real, zero, or imaginary.

The reality of the ordinate depends upon the sign of the quantity under the root

$$(B^2 - 4AC)x^2 + 2(BE - 2CD)x + E^2 - 4CF$$

$$\text{or } Px^2 + Qx + R.$$

Now, writing $\frac{1}{z}$ for x, this expression becomes

$$\frac{P + Qz + Rz^2}{z^2},$$

the sign of which fraction is the same as that of its numerator, since z^2 is essentially positive; but, by diminishing z, we shall arrive at length at some value of z, for which, and for all smaller values, whether positive or negative, $Qz + Rz^2$ is less in magnitude than P, *i. e.* as z decreases beyond this value, the expression will always be of the same sign as P; hence, since $x = \frac{1}{z}$, *there* *is some value of* x, *for which, and for all* greater *values, whether positive or negative, the sign of the expression*

$$Px^2 + Qx + R$$

is the same as the sign of P.

Hence we gain the following results:

I. When P, or $B^2 - 4AC$, is *negative*, there will be values of x beyond which the ordinate y will be always imaginary, whether we take x positive or negative. The curve will then be limited both in the positive and negative directions of x. A curve of this class is called an *Ellipse*. It will be seen hereafter that the *Circle* belongs to this class of curves.

II. When $B^2 - 4AC$ is *positive*, there will be values of x both positive and negative, beyond which the ordinate y will be always real. The curve will then extend indefinitely both in the positive and negative directions of x. A curve of this class is called an *Hyperbola*. In this case if the quantity under the root be a complete square, the equation represents two straight lines (Art. 56) which cut one another; this is, evidently, like the hyperbola, a locus unlimited in every direction.

III. When $B^2 - 4AC$ is $= 0$, the polynomial loses its first term. Then, if $BE - 2CD$ be a positive quantity, there will evidently be some positive value of x, for which, and for all higher values, the quantity under the radical is positive, and the ordinate real; but, if we take x negative, there will be some value of x, for which, and for all higher values, the quantity is negative, and the ordinate imaginary. Exactly the contrary will be the case if $BE - 2CD$ be a negative quantity. Hence, in these two cases, the curve will extend indefinitely *either* in the positive or negative direction of x, and will be limited in the other direction. A curve of this class is called a *Parabola*. If $BE - 2CD = 0$, at the same time that $B^2 - 4AC = 0$, the above reasoning is evidently inapplicable, and the equation will not represent a curve of the above class; it will, in fact, (Art. 56) represent two parallel straight lines. The student will observe that, in case III, the first three terms of the general equation form a complete square.

102. The reasoning of the previous Articles depends on the possibility of solving the equation as a quadratic in y, that is, we have supposed that C is not $= 0$ in the equation. If we solve with regard to x, we have

$$x = \frac{-(By+D) \pm \sqrt{(B^2 - 4AC)y^2 + 2(BD - 2AE)y + D^2 - 4AF}}{2A},$$

which will give us the same conditions for classifying the curves as before, and shews us that the parabola will be indefinitely extended in the positive or negative direction of the axis of y, according as $BD - 2AE$ is positive or negative*.

103. If A or $C = 0$ in the general equation, we may solve with regard to that variable, the coefficient of whose second power does not vanish; in each of these cases the curve will be an hyperbola, for $B^2 - 4AC$ reduces to B^2, and is always positive.

104. If A and C each $= 0$, the equation is of the form

$$Bxy + Dx + Ey + F = 0,$$

which represents an hyperbola, since any value of one variable, however large, or of whatever sign, gives a real value of the other. If, however, this equation can be resolved into two possible factors of the first degree, it represents two straight lines.

105. If $B = 0$ at the same time that A or $C = 0$, $B^2 - 4AC$ becomes $= 0$, and the curve is a parabola.

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* It is shewn (Art. 119), that the conditions, $B^2 - 4AC = 0$, $BE - 2CD = 0$, make also $BD - 2AE = 0$.

106. We may then divide the curves of the second order into three distinct classes:

I. Curves limited in every direction; $B^2 - 4AC < 0$.

II. Curves unlimited in every direction; $B^2 - 4AC > 0$.

III. Curves limited in one direction, and unlimited in the opposite direction. $\Big\}$ $B^2 - 4AC = 0$.

The forms of the curves will be seen hereafter to be as in the figures.

Fig. I.

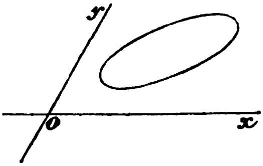


Fig. II.

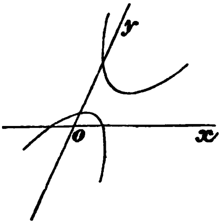
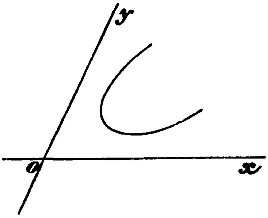


Fig. III.



Ex. 1. Let the proposed equation be

$$3x^2 + 2xy + 3y^2 - 16y + 23 = 0;$$

here $B = 2$, $A = 3$, $C = 3$, and therefore $B^2 - 4AC = -32$, or the curve is an ellipse.

Ex. 2. Let the equation be

$$y^2 - 10xy + x^2 + y + x + 1 = 0;$$

here $B = -10$, $A = 1$, $C = 1$, and $B^2 - 4AC = 96$, or the curve is an hyperbola.

Ex. 3. Let the equation be

$$y^2 - 2xy + x^2 + y - 6x + 9 = 0;$$

here $B = -2$, $A = 1$, $C = 1$, and $B^2 - 4AC = 0$, or the curve is a parabola.

Ex. 4. $$4x^2 + 8xy + 4y^2 + 3x + 3y = 0;$$

here $B^2 - AC = 0$, but also $BE - 2CD = 0$; hence, the equation represents two parallel straight lines, whose equations are, by the method of Art. 56,

$$y + x = 0, \qquad 4y + 4x + 3 = 0.$$

Ex. 5. $$12xy + 8x - 27y - 18 = 0;$$

here $B = 12$, $A = 0$, $C = 0$, and $B^2 - 4AC = 144$; but writing the equation in the form

$$xy + \tfrac{2}{3}x - \tfrac{9}{4}y - \tfrac{3}{2} = 0,$$

$$\text{or } (x - \tfrac{9}{4}) . (y + \tfrac{2}{3}) = 0,$$

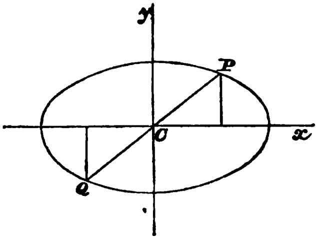
we see that it represents two straight lines

$$x - \tfrac{9}{4} = 0, \qquad y + \tfrac{2}{3} = 0.$$

107. Having divided the loci represented by the equation of the second degree into three classes, we shall now proceed to shew how, by a proper selection of origin and axes, we may reduce the general equation to a form which will enable us, in each case, to trace the curve and deduce its most remarkable properties.

If a point be so situated with regard to a curve, that all chords of the curve, drawn through the point, are bisected in it, the point is called the *centre* of the curve.

Suppose C to be the centre of a curve, and the chord PCQ to meet the curve in P and Q; then $CP = CQ$, and, if we take C as the origin of co-ordinates, it is evident that the co-ordinates of Q are the same as those of P, but with opposite signs; *i. e.* if the co-ordinates of P be x', y', the



co-ordinates of Q are $-x'$, $-y'$, and this is true whatever be the position of P. Hence, the centre being origin, for every point $(x'y')$, whose co-ordinates satisfy the equation, there will be a corresponding point $(-x', -y')$ whose co-ordinates also satisfy the equation.

From this it follows that, when the centre is the origin, the equation will not be altered by writing $-x$, $-y$ for x and y; now this can only be the case when the terms Dx and Ey vanish from the general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0;$$

and, therefore, if we select the centre of a curve for origin, we shall simplify the equation by getting rid of these two terms.

108. In order to find the centre of the curve, we must transfer the origin to a point $(x'y')$, and then observe what values of x', y', make the new D and the new E vanish. These values will be the co-ordinates of the centre with reference to the original axes.

Writing (Art. 59) $x + x'$ for x, $y + y'$ for y in the general equation, we have

$$A(x+x')^2+B(x+x')(y+y')+C(y+y')^2+D(x+x')+E(y+y')+F=0.$$

Arranging the equation according to the powers of the variables, we find that the coefficients of x^2, xy, y^2 will be, as before, A, B, C, and that

$$\text{the new } D = 2Ax' + By' + D;$$

$$\text{the new } E = 2Cy' + Bx' + E;$$

$$\text{the new } F = Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F.$$

Hence, the co-ordinates of the centre, with reference to the original axes, will be determined by the equations

$$2Ax' + By' + D = 0,$$

$$2Cy' + Bx' + E = 0.$$

These two equations will, in general, give one and only one value of x' and y'; hence, *curves of the second degree have in general* one, *and* only *one*, *centre.*

Its co-ordinates are found, by solving the above equations, to be

$$x' = -\frac{BE - 2CD}{B^2 - 4AC}, \quad y' = -\frac{BD - 2AE}{B^2 - 4AC}.$$

109. In the ellipse and hyperbola, $B^2 - 4AC$ is always finite (Art. 101); but in the parabola $B^2 - 4AC = 0$, and the co-ordinates of the centre become infinite. The ellipse and hyperbola are hence often classed together as central curves, while the parabola is called a non-central curve.

110. We see by the last Article that the equation to the ellipse and hyperbola may, by taking the centre of the curve as origin, be reduced to the form

$$Ax^2 + Bxy + Cy^2 + F = 0;$$

we next proceed to inquire whether, by any change in the direction of the axes, we can get rid of the term involving xy, as it will be seen hereafter that this will greatly facilitate our inquiries into the form and properties of the curve. Now it is manifest that, if we can so transform the axes as to get rid of the term involving xy, the equation will be left in the form

$$Ax^2 + Cy^2 + F = 0,$$

where, if any value be given to one of the variables, the other will have two equal values with opposite signs; hence, in this case, each axis will bisect all chords parallel to the other.

111. We have not hitherto supposed the axes necessarily rectangular, though the generality of our reasoning would not have been affected by such a supposition, since, if they were oblique, by transforming them to rectangular axes, we should have obtained an equation of the same degree, and as general as the one we have assumed.

We shall now, for the sake of simplicity, suppose such a change, if necessary, to have been made, and the axes to be rectangular; then, if we turn them round through any angle θ, we must (Art. 60) substitute for x, $x\cos\theta - y\sin\theta$; and for y, $x\sin\theta + y\cos\theta$, in the equation

$$Ax^2 + Bxy + Cy^2 + F = 0.$$

Substituting, and arranging the terms, we shall have

the new $A = A\cos^2\theta + B\cos\theta . \sin\theta + C\sin^2\theta$;

orm. the new $B = 2C\sin\theta . \cos\theta + B(\cos^2\theta - \sin^2\theta) - 2A\sin\theta . \cos\theta$;

49, 50 the new $C = A\sin^2\theta - B\cos\theta . \sin\theta + C\cos^2\theta$;

trigon. the new $F = F$.

Now, if we put the new $B = 0$, we obtain

$$\tan 2\theta = \frac{B}{A - C},$$

an equation from which we may determine the angle θ through which the axes must be moved in order that the term involving xy may vanish.

As the tangent of an angle may have any magnitude, it follows that this equation will always give real values for 2θ; and if we denote by $2a$ that value of 2θ which lies between zero and π, then the positive values of 2θ are

$$2a, \quad \pi + 2a, \quad 2\pi + 2a, \quad 3\pi + 2a, \quad \&c.;$$

consequently, as θ lies between zero and 2π, there are 4 values of θ, viz.

$$a, \quad \frac{\pi}{2} + a, \quad \pi + a, \quad \frac{3\pi}{2} + a,$$

the two former of which determine two lines at right angles to one another, and the two latter the prolongations of these lines; so that, if we take one of these lines for the axis of x, the other will be the axis of y. Hence, there exists one system of rectangular axes, and one only, about which the curve is so situated that each axis bisects all chords parallel to the other. *These axes are called* the axes *of the curve.* We shall see hereafter that their position is that of the figures.

Fig. I.

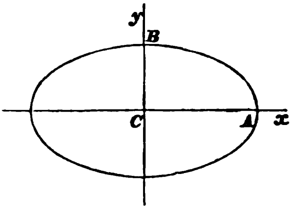
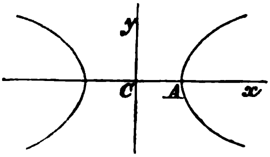


Fig. II.



Cor. If $B = 0$ in the general equation, we have

$$Ax^2 + Cy^2 + Dx + Ey + F = 0, \ldots\ldots (1)$$

which may, evidently, be written in the form

$$A(x - P)^2 + C(y - Q)^2 + R = 0, \ldots\ldots (2)$$

and, by transforming (2) to a point (P, Q), we obtain the equation in the form

$$Ax^2 + Cy^2 + R = 0,$$

and the axes, supposed rectangular, are the axes of the curve. If we do *not* suppose them rectangular, the above reasoning shews that, if the equation be in the form (1), we may transpose the origin to such a point that each axis shall bisect all chords parallel to the other.

112. As the form of the curves has been shewn to depend upon the sign of $B^2 - 4AC$, it is important to shew that this quantity does not change when the axes are moved through any angle θ. We have already (Art. 108) shewn that it does not change for a transfer of the origin. Denoting the new coefficients by A', B', C', we have, from the equations of Art. 111, rig. Form. 51 & 52

$$2A' = (A + C) + (A - C)\cos 2\theta + B\sin 2\theta,$$

$$B' = B\cos 2\theta - (A - C)\sin 2\theta,$$

$$2C' = (A + C) - (A - C)\cos 2\theta - B\sin 2\theta.$$

Hence, $A' + C' = A + C,\ldots\ldots(1)$

and
$$B'^2 - 4A'C' = \{B\cos 2\theta - (A - C)\sin 2\theta\}^2 - (A + C)^2 + \{(A - C)\cos 2\theta + B\sin 2\theta\}^2$$
$$= (A - C)^2 + B^2 - (A + C)^2$$

or $B'^2 - 4A'C' = B^2 - 4AC\ldots\ldots(2)$

113. As the term involving xy has now vanished, the equation to central curves referred to their axes may be written in the form

$$Ax^2 + Cy^2 = F,$$

where the constants A, C, and F have been determined from the formulæ given above (Art. 111) for the transformation of the co-ordinates. We see that, in order to make $B^2 - 4AC$ positive, since $B = 0$, A and C must be of different signs, and in order to make $B^2 - 4AC$ negative, A and C must be of the same sign. Hence, the ellipse will be represented by the equation

$$Ax^2 + Cy^2 = F,$$

and the hyperbola by $Ax^2 - Cy^2 = F$,

where A and C are positive quantities.

In the case of the ellipse, F must be positive, for, when F becomes $= 0$, the curve becomes indefinitely small, since the equation can then be satisfied by no finite values of x and y. When F becomes negative, the ellipse becomes imaginary.

In the case of the hyperbola, F may be positive or negative; but it will not be necessary to take this into consideration, since the equation

$$Cy^2 - Ax^2 = F$$

will represent a curve of the same form; the axis of x in the former equation having the same position with reference to the curve as the axis of y in the latter. If $F = 0$, the hyperbola becomes two straight lines represented by the equations

$$y = \pm \sqrt{\frac{A}{C}}\,x.$$

114. The equation to the ellipse may be written in the following convenient form:

Let the intercept CA made by the ellipse on the axis of x be $= a$, then a is found by making $y = 0$, $x = a$, in the equation

$$Ax^2 + Cy^2 = F,$$

whence

$$Aa^2 = F, \quad \text{and } a^2 = \frac{F}{A}.$$

In like manner, if b be the intercept CB on the axis of y, $b^2 = \dfrac{F}{C}$. Hence the equation of the ellipse

$$\frac{A}{F}x^2 + \frac{C}{F}y^2 = 1$$

may be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

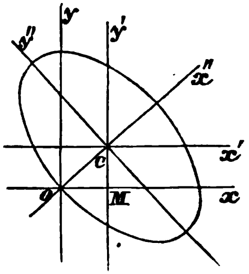
The equation to the hyperbola, which we saw only differs from that of the ellipse in the sign of the coefficient of y^2, may be written in the corresponding form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$$

though in this case we have from the equation ($Ax^2 - Cy^2 = F$) the values $a^2 = \dfrac{F}{A}$, $b^2 = -\dfrac{F}{C}$; so that the intercept on the axis of y is an imaginary quantity, or the curve does not meet that axis.

It may be observed that the criterion by which the curves are distinguished, viz., the sign of B^2-4AC, where A, B, C are the coefficients of x^2, xy, y^2 respectively, holds, under whatever form the equation may be put by transformation; for the nature of the curve cannot be altered by transforming the co-ordinates, and the criterion proved for the most general form of the equation is necessarily true for all particular forms that it may assume.

115. An example will make the preceding Articles clearer.



Let the proposed equation be

$$5y^2 + 5x^2 + 2xy - 12x - 12y = 0; \ldots (1)$$

here $B^2 - 4AC = -96$, or the curve is an ellipse. Now, in order to transfer the origin to the centre of the curve, we write $x + x'$ for x, and $y + y'$ for y, and the equation becomes

$$\left.\begin{array}{l} 5y^2 + 5x^2 + 2(5x' + y' - 6)x + 2(5y' + x' - 6)y \\ + 2xy + 5x'^2 + 5y'^2 + 2x'y' - 12x' - 12y' \end{array}\right\} = 0, \ldots (2)$$

hence, equating the coefficients of x and y to zero, we have, to determine x' and y', the co-ordinates of the centre of the curve,

$$5x' + y' - 6 = 0,$$
$$5y' + x' - 6 = 0;$$

which give $x' = y' = 1$; hence, if we transfer the origin to a point C, so that $OM = MC = 1$, C will be the centre, and the equation to the curve, referred to new axes, Cx', Cy', parallel

$\tan(2\theta) = \frac{168}{95}$ $\frac{2\tan\theta}{1-\tan^2\theta} = \frac{168}{95}$ $95\tan\theta = 84 - 84\tan^2\theta$
$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$ $84\tan^2\theta + 95\tan\theta = 84$
$\tan\theta = 7/12$

to the old ones, obtained by putting $x' = y' = 1$ in equation (2), is

$$5y^2 + 5x^2 + 2xy - 12 = 0. \quad . \quad . \quad . \quad . \quad (3)$$

Next, retaining C as origin, we must find what angle the axes must be turned through, in order that the term containing xy may vanish, *i.e.* in order that each axis may bisect the chords parallel to the other. Writing $x\cos\theta - y\sin\theta$ for x, and $x\sin\theta + y\cos\theta$ for y, in equation (3), we obtain an equation

$$Ax^2 + Bxy + Cy^2 + F = 0, \quad . \quad . \quad . \quad . \quad (4)$$

$$\begin{aligned} \text{where } A &= 5(\sin^2\theta + \cos^2\theta) + 2\sin\theta\cos\theta = 5 + \sin 2\theta, \\ B &= 2(\cos^2\theta - \sin^2\theta) = 2\cos 2\theta, \\ C &= 5(\sin^2\theta + \cos^2\theta) - 2\sin\theta\cos\theta = 5 - \sin 2\theta, \\ F &= -12; \end{aligned}$$

hence, that B may vanish, $\cos 2\theta$ must $= 0$, or $2\theta = 90^\circ$, $\theta = 45^\circ$.

Hence, taking new rectangular axes, Cx'', Cy'', where

$$x''Cx' = 45^\circ,$$

the equation to the ellipse referred to Cx'', Cy'', is

$$3x^2 + 2y^2 = 6,$$

obtained by substituting for θ in equation (4). As the original equation (1) contained no absolute term, it is evident (Art. 77), that O is a point on the curve. Its position with regard to its new axes will be seen hereafter to be that of the figure. The equation may be written $\frac{x^2}{2} + \frac{y^2}{3} = 1$, where (Art. 114) 2 and 3 are the squares of the portions of the axes of x and y respectively, intercepted by the curve, as may be seen by making $y = 0$ and $x = 0$ in the equation.

116. We saw, in Art. 109, that, when $B^2 - 4AC = 0$, in the general equation,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

$\sec^2 = 1 + \tan^2$ $\sin = \tan \times \cos$,
$\cos = \frac{1}{\sec}$.

the co-ordinates of the centre of the curve become infinite. Hence, in the parabola, we cannot, as in the ellipse and hyperbola, destroy the terms involving x and y, by removing the origin to the centre of the curve; but if, as in Art. 111, we move the axes through an angle θ, the new B will vanish by the same conditions as in that Article, viz.

$$\tan 2\theta = \frac{B}{A - C}.$$

But the same condition that makes the new B vanish will also make the new A or the new C vanish, since (Art. 112) $B^2 = 4AC$ in the new equation as well as in the old, and the equation will assume the form

$$Cy^2 + Dx + Ey + F = 0,$$

where we have supposed the new A to vanish, since, if we supposed the new C to vanish, the equation would manifestly not represent a different form of curve, but one having a situation, with regard to the axes of x and y respectively, similar to the situation of the supposed curve with regard to the axes of y and x respectively.

Dividing by C, we have

$$y^2 + \frac{E}{C}y + \frac{D}{C}x + \frac{F}{C} = 0;$$

$$\text{or } y^2 + \frac{E}{C}y + \frac{E^2}{4C^2} + \frac{D}{C}\left(x + \frac{F}{D} - \frac{E^2}{4DC}\right) = 0,$$

which equation may be written in the form

$$(y - P)^2 = L(x - R),$$

where $P = -\dfrac{E}{2C}$, $L = -\dfrac{D}{C}$, $R = \dfrac{E^2}{4DC} - \dfrac{F}{D}$.

If now the origin be transformed to a point whose co-ordinates are $x = R$, $y = P$, we shall have, writing $x + R$ for x. $y + P$ for P,

$$y^2 = Lx,$$

as the equation to the parabola in its simplest form.

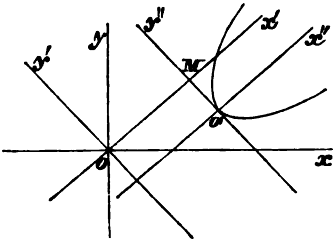
As in Art. 111, it will be seen that there is one system of rectangular axes, and one only, which gives the equation to the parabola in this form.

The axis of x is now called the *axis* of the parabola, and the point where it meets the curve, *i. e.* the origin of co-ordinates, is called the vertex.

117. An example will render the preceding Article clearer. Let the proposed equation be

$$x^2 - 2xy + y^2 - 8x + 16 = 0 \; ; \quad . \quad . \quad . \quad . \quad (1)$$

here $B^2 - 4AC = 0$, or the curve is a parabola, and we cannot remove the origin to the centre; we must therefore find out what angle the axes must be turned through, in order that the term involving xy may vanish.



Writing $x \cos\theta - y \sin\theta$ for x,

and $x \sin\theta + y \cos\theta$ for y,

in equation (1) we have an equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \; ; \quad . \quad . \quad . \quad (2)$$

where $A = 1 - \sin 2\theta$, $B = -2 (\cos^2\theta - \sin^2\theta)$, $C = 1 + \sin 2\theta$, $D = -8 \cos\theta$, $E = 8 \sin\theta$, $F = 16$: hence, in order that B may vanish, we have $\cos\theta = \sin\theta$, or $\theta = 45^0$, and the equation to the parabola, referred to Ox', Oy' as new rectangular axes, where $x'Ox = 45^0$, will be found by making $\theta = 45^0$ in equation (2), to be

$$y^2 + 2\sqrt{2}y - 2\sqrt{2}x + 8 = 0 \; ;$$

$$\text{or}\quad (y+\sqrt{2})^2 = 2\sqrt{2}\left(x - \frac{3}{\sqrt{2}}\right). \quad . \quad . \quad . \quad (3)$$

If we now transfer the origin to a point O' whose co-ordinates are $OM = x = \frac{3}{\sqrt{2}}$, $O'M = y = -\sqrt{2}$, the equation to the parabola, referred to the rectangular axes $O'x''$, $O'y''$, parallel to Ox', Oy', will be, writing $y - \sqrt{2}$ for y and $x + \frac{3}{\sqrt{2}}$ for x in (3),

$$y^2 = 2\sqrt{2}x.$$

We shall see hereafter that its position with regard to the new axes is that of the figure.

118. It will be observed that, in the case of central curves, we have in the reduced equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

two disposable constants, viz. the two axes of the curve. In the general equation with which we started, there were five, and the reader must bear in mind in what manner three have been disposed of, viz. the abscissa and ordinate of the centre (Art. 108), and the inclination of the axis major to the axis of x (Art. 111).

In the case of the parabola we start with four disposable constants, since the relation $B^2 = 4AC$ diminishes the number by one. We then dispose of the abscissa and ordinate of the vertex, and the inclination of the axis of the parabola to the axis of x, and have one left in the reduced equation, viz. the quantity L, the geometrical meaning of which will be hereafter explained.

The condition that a line should pass through a given point $(x'y')$ is only one geometrical condition, since, generally, every line has an abscissa x' and an ordinate y', and the condition is simply that these should belong to the *same* point. That the centre or vertex of a conic should be in a given point is two conditions. The abscissa of the point limits the centre to be in one straight line, and the ordinate limits it to be in another.

119. If in the values of the co-ordinates of the centre, found in Art. 108, we have $BE - 2CD = 0$, at the same time that $B^2 - 4AC = 0$, we also have

$$B^2 = 4AC = 4A\frac{BE}{2D},$$
$$\text{or } BD - 2AE = 0,$$

and both the co-ordinates of the centre become $= \frac{0}{0}$, or indeterminate.

The two equations

$$2Ax + By + D = 0,$$
$$2Cy + Bx + E = 0,$$

will now represent two straight lines that coincide, as may be seen by multiplying the former by E and the latter by D: hence, there are an indefinite number of centres, all situated in that line. The proposed equation, with the above relation between its coefficients, no longer represents a parabola, but (Art. 101, III), two parallel straight lines, between which the above-mentioned line lies evenly, forming a line of centres. An instance of this is given Art. 106, Ex. 4.

120. We proved (Art. 56) that the condition that the general equation of the second degree should represent two straight lines was that the quantity under the root, when the equation had been solved as a quadratic in one of the variables, should be a complete square. By referring to the solution in Art. 101, we shall see that this gives, as the general condition,

$$(B^2-4AC)(E^2-4CF)-(BE-2CD)^2=0, \quad . \quad . \quad (1)$$

or, simplifying, we have

$$FB^2-4ACF-BED+CD^2+AE^2=0. \quad . \quad . \quad (2)$$

If equation (2) be doubled, it may be written

$$2F(B^2-4AC)-D(BE-2CD)-E(BD-2AE)=0, \ . \quad (3)$$

a result which may easily be seen to agree with the conditions of the Articles of the present Section.

EXAMPLES VI.

In working the following examples, the student should illustrate each analytical transformation by a figure. It will perhaps be better to read Arts. 121—131, 193—196, before attempting them.

Transform the following equations:

1. $1+2x+3y^2=0$ to $y^2=-\frac{2}{3}x$.
2. $3x^2+2y^2-2x+y-1=0$ to $72x^2+48y^2=35$.
3. $y^2-2xy+x^2-8x+16=0$ to $y^2=2\sqrt{2}x$.
4. $5(y^2+x^2)+2xy-12(x+y)=0$ to $3x^2+2y^2=6$.
5. $3x^2+2xy+3y^2-16y+23=0$ to $4x^2+2y^2=1$.
6. $y^2-10xy+x^2+y+x+1=0$ to $48y^2-32x^2=-9$.
7. $y^2-2xy+x^2-6y-6x+9=0$ to $y^2=3\sqrt{2}x$.
8. $y^2+xy+x^2+y+x-5=0$ to $9x^2+3y^2=32$.
9. $y^2-3xy+x^2+1=0$, the axes being inclined at an angle of 60^0, to $x^2-15y^2=3$, the axes being rectangular, and the new axis of x inclined to the old at an angle of 30^0.
10. $y^2-2xy-x^2+2=0$ to $y^2-x^2+\sqrt{2}=0$.
11. $y^2-x^2-y=0$ to $4x^2-4y^2+1=0$.
12. $7y^2+16xy+16x^2+32y+64x+28$, the axes being inclined at an angle of 60^0, to $y^2+4x^2=9$, the axes being rectangular, and the axis of x remaining the same.

SECTION VIII.

Central Conic Sections, referred to their axes.*

121. THE figure of the curves may now be deduced from the simple form to which we have reduced their equations. We will commence with the ellipse, and, since we may choose whichever axis we please for axis of x, we shall suppose that we have chosen the axes so that a may be greater than b.

122. *To find the polar equation to the ellipse; the centre being the pole.*

Writing (Art. 51) $\rho\cos\theta$ for x, and $\rho\sin\theta$ for y in the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have

$$\frac{1}{\rho^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2},$$

$$\text{or } \rho^2=\frac{a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta},$$

which is the required equation. It may be written

$$\rho^2=\frac{a^2b^2}{a^2-(a^2-b^2)\cos^2\theta}=\frac{a^2b^2}{b^2+(a^2-b^2)\sin^2\theta},$$

and it will appear hereafter that it is convenient to use the abbreviation $\frac{a^2 - b^2}{a^2} = e^2$. Hence, dividing numerator and denominator by a^2, we have

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* We use the term 'conic section' for a 'curve of the second degree,' because we shall prove hereafter that the section made by any plane in a cone standing on a circular base is a curve of the second degree; and, conversely, that all curves of the second degree may be considered as conic sections.

$$\rho^2 = \frac{b^2}{1 - e^2 \cos^2\theta},$$

the form most commonly used.

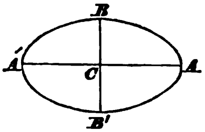
123. This equation will be found the most convenient for tracing the ellipse*. The least value that $b^2 + (a^2 - b^2)\sin^2\theta$ can have, is when $\theta = 0$, therefore, since

$$\rho^2 = \frac{a^2b^2}{b^2 + (a^2 - b^2)\sin^2\theta},$$

the *greatest* value of ρ is the intercept on the axis of x, and is $= a$.

Again, the greatest value of $b^2 + (a^2 - b^2)\sin^2\theta$ is, when $\sin\theta = 1$, or $\theta = 90^0$; hence, the *least* value of ρ is the intercept on the axis of y, and is $= b$. The greatest line then that can be drawn through the centre is the axis of x, and the least line, the axis of y.

From this property, these lines, $A'A$ ($= 2a$) and $B'B$ ($= 2b$) are called the axis *major* and the axis *minor* of the curve. It is plain that the smaller θ is, the greater ρ will be; hence, *the nearer any radius vector is to the axis major, the greater it will be.* The form of the curve will therefore be that of the figure. The points A', A are called the vertices.



124. We obtain the same value for ρ, whether we suppose $\theta = \alpha$, or $\theta = -\alpha$. Hence, *Two radii vectores which make equal angles with the axis will be equal.* And it is easy to shew that the converse of this theorem is true.

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* The Articles on the figure of the curves are taken chiefly from Salmon's *Conic Sections.*

125. The figure of the ellipse may also be seen from the following construction.

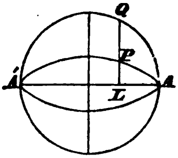
Solving the equation to the ellipse for y, we have

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}.$$

Now, if we describe a concentric circle with the radius a, its equation will be

$$y = \pm \sqrt{a^2 - x^2}.$$

Hence, *if a circle be described on the axis major, and on each ordinate* LQ *a point* P *be taken, such that* LP *may be to* LQ *in the constant ratio* b : a, *then the locus of* P *will be the required ellipse.*



Hence, the circle described on the axis major lies wholly *without* the curve.

We might, in like manner, construct the ellipse, by describing a circle on the axis minor, and *increasing* each ordinate in the constant ratio $a : b$.

Hence, the circle described on the axis minor lies wholly *within* the curve.

We see, also, that the equation to the circle is the particular form which the equation to the ellipse assumes when $b = a$.

126. *To find the equation to the ellipse, when one of the vertices is origin, the direction of the axes being the same as before.*

The problem is, to transfer the origin of co-ordinates to the point A' $(-a0)$; hence, writing $x - a$ for x in the equation to the ellipse,

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

we have

$$y^2 = \frac{b^2}{a^2}(2ax - x^2),$$

for the equation to the ellipse when the origin is the vertex A'.

127. We shall now investigate the figure of the hyperbola from its equation,

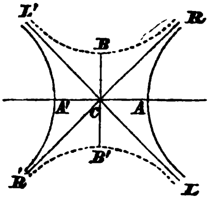
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The intercept on the axis of x is evidently $= \pm a$, but that on the axis of y, being found from the equation $\frac{y^2}{b^2} = -1$, is imaginary; the axis of y then does not meet the curve in real points.

If, however, we take an hyperbola whose equation is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1,$$

the axis of y will meet this curve at a distance $= \pm b$ from the origin, and the axis of x will not meet it in real points. This (for reasons given hereafter) is called the *conjugate hyperbola*, and possesses properties which will be of use to us in considering those of the original curve. We shall then call the distance $A'A$ ($= 2a$), between the two vertices, or points where the curve meets the axis of x, the *transverse axis*, and we shall call the distance $B'B$ between the two points where the conjugate hyperbola meets the axis of y, the *conjugate* axis. For we have chosen as axis of x that which meets the hyperbola in real points, and are, therefore, not entitled to assume a greater than b, so that the terms, axis major and axis minor would not here be applicable.



The conjugate hyperbola will evidently be an hyperbola, whose transverse and conjugate axes correspond to the conjugate and transverse axes of the original curve.

128. *To find the polar equation to the hyperbola, centre pole.*

Transforming the equation to the hyperbola to polar coordinates as in the case of the ellipse, we get

$$\rho^2 = \frac{a^2b^2}{b^2\cos^2\theta - a^2\sin^2\theta} = \frac{a^2b^2}{b^2 - (a^2+b^2)\sin^2\theta} = \frac{a^2b^2}{(a^2+b^2)\cos^2\theta - a^2}.$$

Since formulæ concerning the ellipse are altered to the corresponding formulæ for the hyperbola by changing the sign of b^2, we must, in this case, use the abbreviation e^2 for $\frac{a^2+b^2}{a^2}$. Dividing numerator and denominator by a^2 we have for the polar equation to the hyperbola, the centre being pole,

$$\rho^2 = \frac{b^2}{e^2\cos^2\theta - 1};$$

the form most commonly used.

129. As in the ellipse, the hyperbola may be conveniently traced from this equation. The denominator $b^2 - (a^2 + b^2)\sin^2\theta$ will plainly be greatest when $\theta = 0$, therefore, in that case, ρ will be least; or, *the transverse axis is the least line which can be drawn from the centre to the curve.*

As θ increases ρ continually increases, until

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}} \left(\text{or } \tan\theta = \frac{b}{a}\right),$$

when the denominator of the value of ρ becomes $= 0$, and ρ becomes infinite. After this value of θ, ρ^2 becomes negative, and the radii vectores cease to meet the curve in real points, until again

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}\ \left(\text{or } \tan\theta = -\frac{b}{a}\right),$$

when ρ again becomes infinite. It then decreases regularly as θ increases until θ becomes $= 180^0$, when it again receives its minimum value $= a$. The lower part of the curve evidently corresponds exactly with the upper.

The form of the hyperbola, therefore, is that represented by the dark line in the figure, where the branches LAR, $L'A'R'$ extend to infinity. We have drawn the curves always concave to the axis of x. We shall hereafter (Art. 142) prove the correctness of the figures in this respect.

130. It was shewn that the radii vectores answering to $\tan\theta = \pm\frac{b}{a}$ meet the curve at infinity. These radii vectores, indefinitely produced, are called the *asymptotes* of the curve. They are the lines $R'R$, $L'L$ of the figure, and evidently separate the lines which meet the curve in real points from those which meet it in imaginary, or the whole of the curve is included in the angles RCL, $R'CL'$.

It may readily be seen that the asymptotes of the conjugate hyperbola coincide with those of the original curve, and consequently that it lies wholly within the angles RCL', LCR', and corresponds to the dotted line in the figure.

131. As in the case of the ellipse, we shall find the equation to the hyperbola when the vertex A is taken for origin by writing $x + a$ for x, in the equation

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2).$$

The result is

$$y^2 = \frac{b^2}{a^2}(2ax + x^2).$$

132. If $a = b$, the hyperbola and its conjugate become equal in every respect, and, since

$$\frac{b}{a} = 1 = \tan RCA, \quad \text{(Art. 130)},$$

RCA will $= 45^0$, or the angle between the asymptotes is a right angle. This is called the rectangular or equilateral hyperbola, and is to the hyperbola what the circle is to the ellipse.

133. The quantity e is called the *eccentricity* of the curve:

$$\text{in the ellipse, } e^2 = \frac{a^2 - b^2}{a^2};$$

$$\text{in the hyperbola, } e^2 = \frac{a^2 + b^2}{a^2}.$$

Hence, in the ellipse, e is less than unity; and in the hyperbola greater than unity. When $a = b$ in the ellipse, or the curve becomes a circle, $e = 0$, and, if a remain the same, e increases as b diminishes, or as the curve passes from a circular to an oval form.

134. In the following investigations we shall in most cases consider both the ellipse and hyperbola together, as they have many properties in common, resulting from the similarity of their equations, and the properties of the one may be deduced from those of the other by changing the sign of b^2. We shall, whenever we speak of both curves together, use the signs which apply to the ellipse.

135. *To find the equation to a line touching the curve*

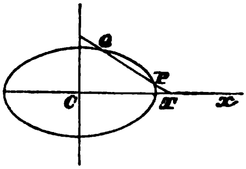
$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right),$$

at the point $(x'y')$*.

---

[*] In Section XII. will be found the method ordinarily employed in finding the equation to the tangent to the ellipse. The reader will find no difficulty in applying the same method to the other curves here treated of.

The method employed is precisely the same as that used in the case of the tangent to the circle (Art. 82), to which the student may refer. We shall simply indicate the steps.



Let the equation to a line cutting the curve in the point $P(x'y')$, be

$$\frac{y-y'}{s} = \frac{x-x'}{c} = l, \quad . \quad . \quad . \quad . \quad (1)$$

then, for the distances (l) between $(x'y')$ and the points of section of the line and curve, we have

$$\frac{(cl+x')^2}{a^2} + \frac{(sl+y')^2}{b^2} = 1\,;$$

$$\text{whence } \left(\frac{c^2}{a^2} + \frac{s^2}{b^2}\right) l^2 + 2\left(\frac{cx'}{a^2} + \frac{sy'}{b^2}\right) l = 0. \quad . \quad . \quad (2)$$

$$\text{Since } \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

for $(x'y')$ is on the curve.

Equation (2) gives us

$$l = 0,$$

as it should, for $(x'y')$ coincides with one of the points of section, and also

$$\left(\frac{c^2}{a^2} + \frac{s^2}{b^2}\right) l + 2\left(\frac{cx'}{a^2} + \frac{sy'}{b^2}\right) = 0,$$

the value of l in which equation is the distance (PQ) of $(x'y')$ from the other point of section Q. If the line be a tangent at $P(x'y')$, this distance vanishes, and we have

$$\frac{cx'}{a^2} + \frac{sy'}{b^2} = 0.$$

Eliminating c and s by means of this equation, and equation (1), we have

$$(x - x')\frac{x'}{a^2} + (y - y')\frac{y'}{b^2} = 0,$$

$$\text{or } \frac{xx'}{a^2} + \frac{yy'}{b^2} - \left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right) = 0,$$

whence $$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

which is the equation to the tangent at the point $(x'y')$, and is easily remembered from its similarity to the equation to the curve*.

If m be the trigonometrical tangent of the angle PTx which the tangent at $(x'y')$ makes with the axis of x, we have

$$m = \frac{s}{c} = -\frac{b^2x'}{a^2y'}.$$

136. Since the formula $m = -\frac{b^2x'}{a^2y'}$ does not change when x', y' are replaced by $-x'$, $-y'$, we see that tangents at the extremities of chords through the centre are parallel. Also at the points B, B', for which $x' = 0$, $y' = \pm b$, $m = 0$, or the tangents are parallel to the axis of x; and at A, A', for which $x = \pm a$, $y = 0$, $m = \infty$, or the tangents are perpendicular to the axis of x.

137. *To find the equation to the tangent in terms of its inclination to the axis of* x.

Proceeding as in (Art. 135), we have, for the equation to the tangent,

$$\frac{y - y'}{s} = \frac{x - x'}{c}, \quad . \quad . \quad . \quad . \quad (1)$$

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* For the form of the equation when the origin is transferred, vide Art. 83.

with the condition

$$\frac{cx'}{a^2} + \frac{sy'}{b^2} = 0; \quad . \quad . \quad . \quad . \quad (2)$$

and we must now eliminate x' and y' instead of s and c:

$$\text{from } (2) \quad \frac{as}{bc} = -\frac{bx'}{ay'};$$

$$\text{whence } \frac{a^2s^2 + b^2c^2}{b^2c^2} = \frac{b^2x'^2 + a^2y'^2}{a^2y'^2} = \frac{a^2b^2}{a^2y'^2}, \quad . \quad . \quad (3)$$

since $(x'y')$ is on the curve, and therefore

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1.$$

From equations (2) and (3), we have

$$y' = \pm \frac{b^2c}{\sqrt{a^2s^2 + b^2c^2}}, \qquad x' = \mp \frac{a^2s}{\sqrt{a^2s^2 + b^2c^2}};$$

whence from (1)

$$\begin{aligned} cy - sx &= cy' - sx' \\ &= \pm\sqrt{a^2s^2 + b^2c^2}, \end{aligned}$$

according as we take the upper or lower signs.

If $\frac{s}{c} = m$, where m is the tangent of the angle which the line makes with the axis of x, the equation becomes

$$y = mx \pm \sqrt{m^2a^2 + b^2}.$$

The double sign refers to the two tangents at the extremities of a chord through the centre, since, by the last Article, these tangents have the same inclination to the axis of x.

The student will have no difficulty in obtaining this equation without introducing the point $(x'y')$, as in the circle, Art. 86.

Cor. Hence, the condition that a line $y = mx + c$, should touch the conic section $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is $c^2 = m^2a^2 + b^2$.

138. If the tangent pass through the centre, we have $\sqrt{m^2a^2 + b^2} = 0$. (Art. 21, Cor. 3.) This gives an impossible value for m in the case of the ellipse, but for the hyperbola we have $m^2a^2 - b^2 = 0$, or $m = \pm\frac{b}{a}$; hence, a tangent to the hyperbola drawn from the centre coincides (Art. 130) with the asymptote, and only meets the curve at an infinite distance. The asymptotes may therefore be considered as lines *touching* the curve at an infinite distance.

139. The result of the last Article may be obtained from the equation to the tangent at any point $(x'y')$, as follows:

The equation to the tangent to the hyperbola at $(x'y')$, is (Art. 135),

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{or } \frac{x}{a^2}\cdot\frac{x'}{y'} - \frac{y}{b^2} = \frac{1}{y'};$$

and when y' is indefinitely large, or the point of contact is indefinitely distant, the right hand member vanishes, $\frac{x'}{y'}$ becomes indeterminate, since it assumes the form $\frac{\infty}{\infty}$, which is, generally, a finite quantity, and the equation becomes

$$\frac{x}{a^2}\cdot\frac{x'}{y'} - \frac{y}{b^2} = 0;$$

hence, the line it represents passes through the centre, and therefore, by similar triangles,

$$\frac{x'}{y'} = \frac{x}{y},$$

and the equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which represents the two asymptotes,

$$\frac{x}{a} + \frac{y}{b} = 0, \qquad \frac{x}{a} - \frac{y}{b} = 0.$$

In the ellipse, the same process would give

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0,$$

which is impossible; hence, in the ellipse, the asymptotes are imaginary *.

140. The student may, perhaps, object to the proof of the last Article, that we began with the equation to *one* straight line, (1), and, on endeavouring to trace its limiting position, consequent on the indefinite increase of x' and y', we have obtained an equation which represents *two* straight lines, one of which is evidently not the limiting position of (1).

---

* Since the asymptote touches the branch RAL (fig. Art. 127) at infinity, it must also, by the symmetry of the figure, touch the branch $R'A'L'$ at infinity. But we saw (Art. 135) that, when we combine the equations to a straight line and the curve, the result is a quadratic which will determine two points of *intersection*, and only two. That is, a straight line can only intersect a curve of the second order in two points, and when these two points coincide, the line is said to touch the curve; hence, a straight line can only touch a curve of the second order in one point, and, consequently, the two points of contact of the asymptote would seem to coincide. This difficulty will be removed when we consider that the asymptote RR' does not really *touch the hyperbola at infinity*, and that this expression is only a short way of saying that, as the distance of the point of contact becomes very great, whether on the branch AR or $A'R'$, the tangent in each case tends to coincide with the line RR'.

But it must be borne in mind that (1) represents, not one, but a class of lines, indefinite in their number, and subject only to the limitation that $(x'y')$ should be on the curve. The suppositions we have made with regard to x' and y' are equally applicable to the whole class. Hence the resulting equation (2) represents the limiting position, not of one tangent but of all of them.

141. *To determine the equations to the tangents to an ellipse or hyperbola which pass through a given point* (x′y′).

The equation to the tangent, is (Art. 137)

$$y - mx = \pm\sqrt{a^2m^2 + b^2}.$$

Since it passes through $(x'y')$, we have

$$(y' - mx')^2 = a^2m^2 + b^2;$$

$$\text{or } m^2 + \frac{2x'y'}{a^2 - x'^2}m + \frac{b^2 - y'^2}{a^2 - x'^2} = 0, \quad . \quad . \quad (1)$$

which equation gives two values of m; let them be m and m'; then the equations of the tangents required are

$$y - y' = m(x - x'), \quad y - y' = m'(x - x'). \quad . \quad . \quad (2)$$

142. The roots of equation (1) (Art 141) are possible and different, equal, or impossible, according as

$$x'^2y'^2 > = < (b^2 - y'^2)(a^2 - x'^2), \quad . \quad . \quad . \quad (3)$$

$$\text{or as } \frac{x'^2}{a^2} + \frac{y'^2}{b^2} > = < 1, \quad . \quad . \quad . \quad (4)$$

and a little consideration will shew that inequality (4) gives the conditions that $(x'y')$ shall be without, on, or within the ellipse. Hence, no tangent can be drawn to the ellipse from within the curve.

To prove the same property for the hyperbola, we must write $-b^2$ for b^2 in (3), and we shall then obtain, instead of inequality (4),

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} <=> 1. \quad . \quad . \quad . \quad . \quad (5)$$

Condition (5) will, evidently, *always* give possible roots when x' does not exceed $\pm a$, but, when x' passes those values, the value of y' must not be less than that of the corresponding ordinate of the curve, since it is that value which makes

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1.$$

Hence, no tangent can be drawn to the hyperbola from within the curve. The above reasoning shews that the curves are, as we have drawn them, always concave to the axis of x; for if this were not the case, it would, evidently, be possible to draw a line from within, which should pass through two coincident points of the curve.

143. *To determine the locus of the intersection of two tangents at right angles to one another.*

If, in Article 141, the two tangents are at right angles, $mm' = -1$. Hence, from equation (1), since mm' is the product of the roots,

$$\frac{b^2 - y'^2}{a^2 - x'^2} = mm' = -1.$$

Hence, $x'^2 + y'^2 = a^2 + b^2$; or the locus of the point x', y', where the tangents intersect, is a circle, whose centre is C and radius $= \sqrt{a^2 + b^2}$.

The corresponding equation for the hyperbola is, of course,

$$x^2 + y^2 = a^2 - b^2,$$

which is a circle, unless b^2 be greater than a^2, in which case the locus is impossible; *i. e.* two tangents cannot be drawn at right angles to one another when b^2 is greater than a^2. In the equilateral hyperbola $b = a$, and the circle is reduced to a point, namely, the origin; hence, only one pair of tangents at right angles to one another can be drawn to the equilateral hyperbola, and those pass through the centre, *i.e.* the asymptotes (Art. 132) which are tangents at an infinite distance.

144. *To find the perpendicular from the centre on the tangent in terms of the angle which it makes with the axis of* x.

If the perpendicular p make an angle α with the axis of x, the equation to the tangent is (Art. 20)

$$x \cos\alpha + y \sin\alpha = p,$$

and hence, by the condition given in Art. 137, Cor.

$$p^2 = a^2 \cos^2\alpha + b^2 \sin^2\alpha,$$

or since $b^2 = a^2 (1 - e^2)$ (Art. 133)

$$p^2 = a^2 (1 - e^2 \sin^2\alpha).$$

Hence the equation to the tangent in the form given in Article 20 is

$$x \cos\alpha + y \sin\alpha - a\sqrt{1 - e^2 \sin^2\alpha} = 0.$$

If we had used the signs suitable to the hyperbola, we should have obtained the same result for that curve.

145. *To find the equation to the normal at any point* (x′y′).

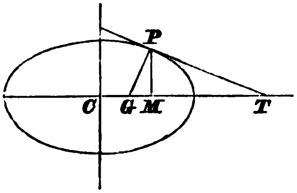
By reasoning similar to that used in the case of the circle (Art. 88), since the normal passes through the point $(x'y')$, and is perpendicular to the tangent, whose equation is

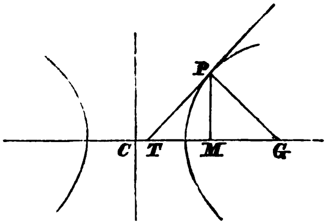
$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

its equation is $y - y' = \dfrac{a^2 y'}{b^2 x'} (x - x')$.

146. The intercepts of the tangent and normal on the axis of x may be found by putting $y = 0$ in their equations.

Let them meet the axis in T and G; then putting $y = 0$ in the equation to the tangent, we have





$$\frac{xx'}{a^2} = 1,$$

$$\text{or } CT = \frac{a^2}{x'};$$

and, in the same manner from the equation to the normal, when $y = 0$, we have

$$x = x'\left(1 - \frac{b^2}{a^2}\right) \text{ or } CG = e^2x'.$$

The portion MT, intercepted on the axis between the tangent and the ordinate of the point of contact is called the *subtangent*, and MG is called the *subnormal*. The length of the subtangent is, in the ellipse,

$$CT - CM = \frac{a^2}{x'} - x' = \frac{a^2 - x'^2}{x'};$$

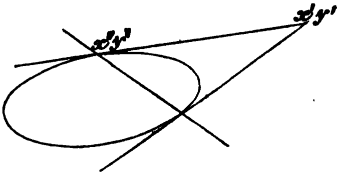
and, in the hyperbola,

$$CM - CT = x' - \frac{a^2}{x'} = \frac{x'^2 - a^2}{x'}.$$

In the same manner the subnormal in the ellipse $= x' - e^2x'$, and, in the hyperbola, $= e^2x' - x'$, or, in both curves, the

$$\text{subnormal} = \frac{b^2}{a^2}x'.$$

147. *To find the equation to the chord joining the points of contact of two tangents drawn from any point* (x′y′).



By the method used (Art. 89) in the case of the circle, if $(x''y'')$ be one of the points of contact, the equation to the tangent at that point is

$$\frac{xx''}{a^2}+\frac{yy''}{b^2}=1;$$

and, since it passes through $(x'y')$, we have

$$\frac{x'x''}{a^2}+\frac{y'y''}{b^2}=1;$$

hence $(x''y'')$ is a point on the line

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1, \ldots\ldots\ldots\ldots\ldots (1)$$

and, similarly, the other point of contact is on this straight line; hence (1) is the required equation to the chord joining the two points of contact.

Cor. Hence, to draw tangents to the curve from any external point $(x'y')$, we have the two equations

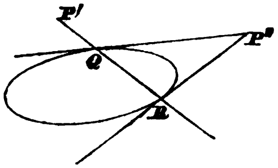
$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1,\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

to determine the co-ordinates of the points of contact.

These equations will always give two points of contact, corresponding to the intersection of the line and curve.

148. *To find the equation to the polar of any point* (x′y′) *with regard to the ellipse or hyperbola.*

Exactly as in the case of the circle (Art. 90), let P' be



the point $(x'y')$ of which the equation to the polar is sought. Draw a chord $P'QR$, and let tangents at Q and R meet in P''; it is required to find the locus of $P''(x''y'')$, as $P'QR$ moves about P'. By Article 147, the equation to $P'QR$ is

$$\frac{xx''}{a^2}+\frac{yy''}{b^2}=1,$$

and since it passes through $P'(x'y')$, we have

$$\frac{x'x''}{a^2}+\frac{y'y''}{b^2}=1;$$

but $(x''y'')$ is *any* point in the locus; hence, the locus is a straight line, defined by the equation

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1,$$

which is the required equation to the polar.

149. We may observe here, as in the circle, that this equation represents

(i) the locus of the intersection of tangents drawn at the extremities of chords through $(x'y')$, *whatever* be the position of the point $(x'y')$;

(ii) the chord joining the points of contact of a pair of tangents drawn from $(x'y')$ to the curve, if $(x'y')$ be without the curve (Art. 147);

(iii) the tangent at $(x'y')$, if the point $(x'y')$ be on the curve (Art. 135).

150. *If any point* (x''y'') *be taken on the polar of* (x'y') *the polar of* (x''y'') *must pass through the point* (x'y').

For the condition that $(x''y'')$ should lie on the polar of $(x'y')$, whose equation is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

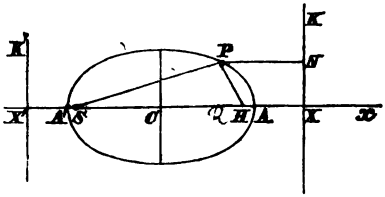
is $$\frac{x''x'}{a^2} + \frac{y''y'}{b^2} = 1,$$

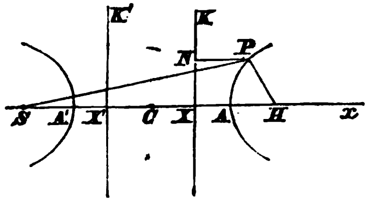
which is also the condition that $(x'y')$ should lie on the line

$$\frac{xx''}{a^2} + \frac{yy''}{b^2} = 1,$$

which is the polar of $(x''y'')$.

151. If, in the equation to the polar of the point $(x'y')$ we make $y' = 0$, the equation becomes $\frac{xx'}{a^2} = 1$, the equation to a straight line parallel to the axis of y; hence, the polar of any point on the axis of x, is parallel to the axis of y; and similarly the polar of any point on the axis of y is parallel to the axis of x.





If we take two points, H, S on the axis of x, so that

in the ellipse $CS = CH = \sqrt{a^2 - b^2} = ae$,

and in the hyperbola $CS = CH = \sqrt{a^2 + b^2} = ae$,

the equations to the polars of these points will be, writing 0, ae and 0, $-ae$ for y', x',

$$x = \frac{a}{e} \text{ for the polar of } H,$$

$$x = -\frac{a}{e} \text{ for the polar of } S.$$

These points are called the *foci*, and their polars are called the *directrices* of the curve.

In the ellipse, since e is less than unity, the foci lie between the centre and the vertex, and the directrices KX, $K'X'$, the polars, respectively, of H and S, lie beyond the vertex. In the hyperbola, since e is greater than unity, the reverse will be the case. It will be seen hereafter, that the focus and its polar, the directrix, possess many remarkable properties in connexion with the curve.

152. *To find the distance of any point in the ellipse from the focus, in terms of the abscissa of the point.*

Since the co-ordinates of the point H are ($x = ae$, $y = 0$,) the square of the distance of any point P, ($x'y'$) from it is (Art. 6),

$$(x' - ae)^2 + y'^2 = x'^2 - 2aex' + a^2e^2 + y'^2;$$

and if P be a point in the curve,

$$y'^2 = \frac{b^2}{a^2}(a^2 - x'^2) = (1 - e^2)(a^2 - x'^2),$$

since $b^2 = a^2 - a^2e^2$. Hence

$$HP^2 = a^2 - 2aex' + e^2x'^2,$$

$$\text{or } HP = a - ex'.$$

We do not notice the value $(ex' - a)$ obtained by giving the negative sign to the square root. For e is less than 1, and x' less than a, hence $ex' - a$ is constantly negative, and need not be taken into consideration, since we are now examining the *magnitude,* not the direction, of the radius vector HP.

Writing $-ae$ for ae in the preceding proof, we have for the distance of P from the other focus

$$SP = a + ex';$$

$$\text{hence } SP + HP = 2a,$$

or *the sum of the distances of any point in an ellipse from the foci is constant, and equal to the axis major.*

153. In the case of the hyperbola, we obtain the same value for HP^2, but, in extracting the root, we must take the value

$$HP = ex' - a,$$

since, in the hyperbola, x' is greater than a, and e is greater than unity, and, consequently, $a - ex'$ is constantly negative. In like manner, we have

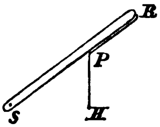
$$SP = ex' + a;$$

$$\text{hence } SP - HP = 2a,$$

or, *in the hyperbola, the* difference *of the focal radii is constant, and equal to the transverse axis.*

154. The property proved in the two last Articles will enable us to describe an ellipse or hyperbola mechanically; for, evidently, if a string SPH be fastened to two points S and H, a pencil P, moved so as to keep the string always stretched, will describe an ellipse of which S and H are the foci, since $SP + HP$ will be a constant quantity.

Also, any portion of an hyperbola may be described by a ruler and cord; for let a ruler SR revolve round S in the plane of the paper, and let a cord be fastened to H, shorter than SR by a given difference c; then a pencil P, which should always keep the string stretched against



SR, would describe an hyperbola; for the difference $SP - PH$ would always equal a constant quantity c, the difference in the lengths of the ruler and cord.

155. *The distance of any point on the curve from the focus is in a constant ratio to its distance from the directrix.*

The equation to the directrix KX is (Art. 151)

$$ex - a = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the length of the perpendicular (fig. Art. 151), from any point P $(x'y')$ in the ellipse on (1) is, (Art. 43), since we do not pass through the line to reach the origin,

$$= -\frac{ex' - a}{e}$$

$$= \frac{1}{e} HP.$$

In the hyperbola, since we do pass through the line, the perpendicular from any point P $(x'y')$, is

$$= \frac{ex' - a}{e}$$

$$= \frac{1}{e} HP.$$

Hence, in both curves, the distance of any point P from the focus is to its distance from the directrix in the constant ratio of e to 1.

In the ellipse $e = \frac{\sqrt{a^2 - b^2}}{a}$ and is less than 1, or the distance from the focus is *less* than the distance from the directrix. In the hyperbola $e = \frac{\sqrt{a^2 + b^2}}{a}$ and is greater than 1, or the distance from the focus is *greater* than the distance from the directrix. It will afterwards be seen that the parabola has a focus, and that the distance of any point from the focus is *equal* to the distance from the directrix.

Hence it is often given as a definition of a conic section, that *it is the locus of a point whose distance from a given point has a fixed ratio to its distance from a given straight line.*

156. The double ordinate through the focus is called the *Latus Rectum.* Putting $x = ae$ in the equation to the curve, we have

$$y^2 = b^2(1 - e^2) = \frac{b^4}{a^2},$$

or $y = \pm \dfrac{b^2}{a}$; hence the latus rectum $= \dfrac{2b^2}{a}$.

157. *To find the polar equation to the ellipse and hyperbola, the focus being the pole.*

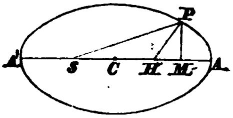
Let $SP = \rho$, angle $PSA = \theta$; then, from Article 152,

$$SP = a + ex';$$

but $x' = CM = SM - SC = \rho \cos\theta - ae$; hence, writing this value for x', we have

$$\rho = a + e\rho \cos\theta - ae^2,$$

$$\text{or } \rho = \frac{a(1 - e^2)}{1 - e\cos\theta}.$$



If we take the focus H in the hyperbola, (fig. Art. 151), we have

$$HP = ex' - a,$$

$$\text{or } \rho = e(ae + \rho\cos\theta) - a,$$

$$\text{whence } \rho = \frac{a(e^2 - 1)}{1 - e\cos\theta}.$$

If we take the focus H in the ellipse, and S in the hyperbola, the point P in the hyperbola being in the left hand branch, we shall obtain the equations

$$\rho = \frac{a(1-e^2)}{1+e\cos\theta} \text{ for the ellipse,}$$

$$\text{and } \rho = \frac{a(e^2-1)}{1+e\cos\theta} \text{ for the hyperbola.}$$

Thus, we see that the positive and negative foci of the ellipse correspond to the negative and positive foci of the hyperbola, the point being taken in that branch of the hyperbola in which the focus lies. We have here supposed ρ positive, but if we take also the negative values of ρ (Art. 50) it is easy to see that each equation will represent both branches of the hyperbola.

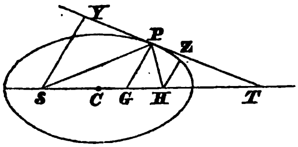
158. *In the ellipse the normal bisects the interior angle between the focal distances, and in the hyperbola the exterior angle; and the focal radii make equal angles with the tangent.*

The equation to HP, since it passes through $(x'y')$, $(ae, 0)$ is

$$y = \frac{y'}{x'-ae}(x-ae), \quad . \quad . \quad (1)$$

and similarly, the equation to SP is

$$y = \frac{y'}{x'+ae}(x+ae). \quad . \quad . \quad (2)$$



Hence, forming the equation to the bisector PG of the angle between (1) and (2), we have, by the rules of Art. 47,

$$\frac{(x'+ae)y-y'(x+ae)}{\{y'^2+(x'+ae)^2\}^{\frac{1}{2}}} = \frac{-(x'-ae)y+y'(x-ae)}{\{y'^2+(x'-ae)^2\}^{\frac{1}{2}}} \quad . \quad . \quad . \quad (3)$$

$= a+ex$, $= a-ex$ p 152

which we might shew, by reduction, to be the equation to the normal at $(x'y')$. This may, however, be proved briefly as follows. The denominators of (3), as in Art. 152, reduce, for the ellipse, to $a+ex'$ and $a-ex'$; hence, to find where (3) meets the axis of x, we have, making $y=0$,

$$\frac{-(x+ae)}{a+ex'}=\frac{x-ae}{a-ex'};$$

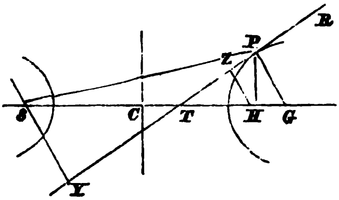
hence $x=CG=e^2x'$, and $\therefore$ (Art. 146) the bisector PG is the normal.

The reasoning for the hyperbola is precisely the same; but the denominators of (3) will now become $ex'+a$ and $ex'-a$, and we shall have, to determine the point T where the bisector PT meets the axis,

$$\frac{-(x+ae)}{ex'+a}=\frac{x-ae}{ex'-a};$$

hence $x=CT=\dfrac{a^2}{x'}$,

and $\therefore$ (Art. 146) the bisector PT is the tangent, and, consequently, the normal PG bisects the exterior angle HPR.



159. *To find the locus of the extremities of perpendiculars dropped from the foci upon the tangent.*

Let SY, HZ be these perpendiculars, then the equation to PT is, taking the fig. of the ellipse,

$$y=mx+\sqrt{m^2a^2+b^2},$$

and since HZ passes through $H\,(\sqrt{a^2-b^2},\,0)$, and is perpendicular to PT, its equation is

$$y=-\frac{1}{m}\left(x-\sqrt{a^2-b^2}\right),$$

between which equations if we eliminate m, we shall obtain the equation to the locus of their intersection. These equations may be written

$$y - mx = \sqrt{m^2a^2 + b^2}, \quad my + x = \sqrt{a^2 - b^2};$$

adding their squares,

$$(x^2 + y^2)(m^2 + 1) = a^2(m^2 + 1),$$

$$\text{or } x^2 + y^2 = a^2,$$

the equation to the locus of Z, which represents a circle on the axis major as diameter.

160. The perpendicular from the point $H(ae, 0)$ on the line

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is, by Art. 43, since in the ellipse we *do not* pass through (1) to reach the origin,

$$= -\frac{\frac{ex'}{a} - 1}{\sqrt{\frac{x'^2}{a^4} + \frac{y'^2}{b^4}}},$$

$$\text{or } HZ = \frac{ab^2(a - ex')}{\sqrt{a^4y'^2 + b^4x'^2}}.$$

But $a^4y'^2 + b^4x'^2 = a^2b^2\left(\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2\right)$

$$= a^2b^2\left\{a^2 - x'^2 + (1 - e^2)x'^2\right\}$$

$$= a^2b^2(a^2 - e^2x'^2),$$

$$\text{and } \therefore \ HZ = b\sqrt{\frac{a - ex'}{a + ex'}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{similarly } SY = b\sqrt{\frac{a + ex'}{a - ex'}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence $SY \, . \, HZ = b^2$. . . . . (4)

In working this problem for the hyperbola, we must remember that, in the case of HZ (fig. Art. 158), we *do* pass through (1) to reach the origin. We mention this because the student would otherwise probably be puzzled by getting $-b^2$ as the right hand member of (4).

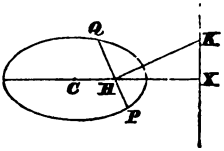
If $SY = p$, $SP = \rho$, equation (3) may be written

$$p^2 = b^2 \frac{\rho}{2a \mp \rho},$$

taking the upper or lower sign, according as the curve is an ellipse or hyperbola.

161. *Any focal chord is perpendicular to the line joining its pole with the focus.*

By definition, the directrix is the polar of the focus, and conversely, by Art. 150, the polar of any point in the directrix passes through the focus. Hence assume the pole of any focal chord PHQ to be a point $K\left(\frac{a}{e}, y'\right)$ on the directrix, then the equation to the chord is (Art. 148)



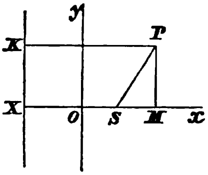
$$\frac{x}{ae} + \frac{yy'}{b^2} = 1,$$

and the equation to a perpendicular HK to this chord through the focus is

$$y = \frac{aey'}{b^2}(x - ae);$$

and when $x = CX = \frac{a}{e}$ in this equation, $y = y' = KX$, or the line which it represents passes through K, the pole of the chord.

162. We have deduced all the above properties of the ellipse and hyperbola from their equations alone; we shall now shew how, conversely, their equations may be deduced from a knowledge of their properties; for example, let it be required *to find the locus of a point* (P), *the distance of which from a given point* (S) *has a constant ratio to its distance from a given straight line* (KX).



Suppose $SP : PK :: e : 1$; draw SX perpendicular to KX, and divide SX in O, so that $SO : OX :: e : 1$; then O will be a point in the locus; take OSx, Oy as axes, and let $OM (= x)$, $MP (= y)$ be the co-ordinates of P; let $OS = d$, and, therefore, $OX = \frac{d}{e}$. Then

$$SM^2 + PM^2 = SP^2,$$

$$\text{or } (x - d)^2 + y^2 = e^2 \,.\, PK^2;$$

$$\therefore \quad = e^2 \left(\frac{d}{e} + x\right)^2;$$

whence we obtain

$$(1 - e^2)\, x^2 + y^2 - 2\,(1 + e)\, dx = 0, \quad . \quad . \quad . \quad (1)$$

an equation of the second degree between x and y.

To find to which of the three classes, mentioned in Article 106, the curve belongs, we may apply the test of that Article, and since B, the coefficient of xy, $= 0$ in this equation, the curve will be an ellipse, hyperbola, or parabola, according as $-4AC$ or $4(e^2 - 1)$ is negative, positive, or zero, for A, C are the coefficients of x^2, y^2 respectively; that is, according as e is less, greater, or equal to unity. This agrees with the results of Article 155.

Hence we see that the simplest form by which the three conic sections may be represented is

$$y^2 = mx + nx^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

when the curve is an ellipse, hyperbola, or parabola, according as n is negative, positive, or zero*.

The point O will plainly be one of the vertices of the curve, S the focus, and KX the directrix.

163. The above proposition may be proved more generally as follows.

Let the given point be $(x'y')$, and the given line

$$Ax + By + C = 0,$$

then, if (xy) be any point on the locus, its distance from $(x'y')$

$$= \{(x - x')^2 + (y - y')^2\}^{\frac{1}{2}},$$

and its distance from the given line is

$$= \frac{Ax + By + C}{\sqrt{(A^2 + B^2)}}.$$

Hence, by the conditions of the problem, we have

$$(x - x')^2 + (y - y')^2 = \frac{e^2}{A^2 + B^2}(Ax + By + C)^2,$$

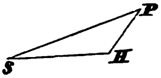
which is an equation of the second degree, and therefore represents a conic section.

164. *To find the locus of a point* P, *the sum of whose distances from two given points* S *and* H *is a constant quantity.*

---

* If a rectangle be constructed on the abscissa OM, having its *erect side* equal to m or $2(1+e)d$, this rectangle $= mx$, and it is evident from (2) that, according as n is negative, positive, or zero, i. e. as e is less, greater, or equal to unity, the square on the ordinate falls short of, exceeds, or equals this rectangle. This defect, excess, or equality are considered by some to have given rise to the names ellipse, hyperbola, and parabola. It is easily seen that m, or $2(1+e)d$ is the same quantity before (Art. 156) defined as the *Latus Rectum.*

Let $SP = \rho$, $HP = \rho'$, $SP + HP = 2a$, $SH = 2c$, angle $PSH = \theta$; then



$$\rho + \rho' = 2a,$$

$$\text{whence } \rho'^2 = \rho^2 - 4a\rho + 4a^2,$$

$$\text{but } \rho'^2 = \rho^2 + 4c^2 - 4c\rho \cos\theta;$$

$$\therefore\ a^2 - a\rho = c^2 - c\rho\cos\theta,$$

$$\text{or } \rho = \frac{c^2 - a^2}{c\cos\theta - a};$$

now, if we write ae for c, where e is less than unity, since $SP + HP > SH$, and therefore $a > c$, the equation becomes

$$\rho = \frac{a(1 - e^2)}{1 - e\cos\theta},$$

which is, evidently, (Art. 157), the equation to an ellipse, whose foci are S and H, and whose eccentricity $= e$.

165. *To find the locus of a point* P, *the difference of whose distances from two given points* S *and* H *is a constant quantity.*

Let $HP = \rho$, $SP = \rho'$, $\pi - PHS = \theta$, then

$$\rho' - \rho = 2a,$$

$$\text{whence } \rho'^2 = \rho^2 + 4a\rho + 4a^2,$$

$$\text{also } \rho'^2 = \rho^2 + 4c^2 + 4c\rho\cos\theta;$$

hence, as before, we have

$$\rho = \frac{a(e^2 - 1)}{1 - e\cos\theta},$$

where e is greater than unity, since SH is greater than $SP - HP$, or $c > a$.

This equation (Art. 157) represents an hyperbola, whose foci are S and H, and whose eccentricity is e.

EXAMPLES VII.

THE following problems are enunciated, some for the ellipse, and some for the hyperbola, though many of them are equally applicable to both curves.

1. If the tangent to an hyperbola, at a point, whose abscissa, measured from the vertex A', is $A'M$, meet the transverse axis in T; $A'M$, $A'A$, $A'T$ will be in harmonical progression.

2. The distance of the centre of an ellipse from a tangent inclined to the major axis at an angle ϕ, is $= a(1 - e^2 \cos^2\phi)^{\frac{1}{2}}$.

3. The distance of the focus of an ellipse from a tangent inclined to the major axis at an angle ϕ is $= a\{e \sin\phi + (1 - e^2 \cos^2\phi)^{\frac{1}{2}}\}$.

4. Find the angle (θ) at which the focal distance SP is inclined to the major axis, when SP is a mean proportional between the semi-axes of an ellipse, when $a = 50$, $b = 30$.

5. If in any hyperbola, three abscissæ be taken in arithmetical progression, the focal distances of the extremities of the ordinates of these points will also be in arithmetical progression.

6. Shew that the equations to the tangents to an ellipse ($3x^2 + y^2 = 3$), inclined at an angle of 45^0 to the axis of x, are $y = x + 2$, $y = x - 2$.

7. If the semi-axes of an ellipse are 5 and 4, find the angle at which CP is inclined to the major axis, when an arithmetic mean between CA and CB.

8. At what point of an ellipse is the angle formed by the focal distances the greatest?

9. If any number of hyperbolas be described, having the same transverse axis, the tangents drawn at the extremities of their latera recta will all pass through one point.

10. If the tangent at any point P in an hyperbola intersect the axis in T, and CP meet the tangent at A in E, ET is parallel to AP.

11. Shew $\tan\dfrac{PSH}{2} \cdot \tan\dfrac{PHS}{2} = \dfrac{1-e}{1+e}$, where P is any point in an ellipse.

12. If *PSp* be any focal chord of an ellipse, L the latus rectum, then $L \, . \, Pp = 4SP \, . \, Sp$.

13. If CA, CB be the semi-axes of an ellipse, shew that, when SBH is a right angle, $CA^2 : CB^2 :: 2 : 1$.

14. Find the condition that the line $\left(\frac{x}{m} + \frac{y}{n} = 1\right)$ should touch the hyperbola $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right)$.

15. The tangent to an ellipse is inclined to the major axis at an angle ϕ; shew that the area included by this tangent and the axes is

$$= \tfrac{1}{2}(a^2 \tan\phi + b^2 \cot\phi).$$

16. The circle described on any radius vector SP of an ellipse as diameter, will touch the circle on the axis major.

17. Find where the tangents from the foot of the directrix will meet the hyperbola, and what angle they will make with the transverse axis.

18. Draw a tangent at the extremity of the latus rectum of an ellipse whose equation is $\frac{x^2}{9a^2} + \frac{y^2}{4a^2} = 1$.

19. A tangent at the extremity of the latus rectum of an hyperbola meets any ordinate PM produced in R; shew that $SP = MR$, where S is the focus through which the latus rectum passes.

20. If PG, PG' be parts of a normal to an hyperbola cut off by the transverse and conjugate axes respectively, prove that $PG : PG' :: b^2 : a^2$.

21. Find the radius of a circle inscribed in a semi-ellipse, touching the axis minor.

22. From the point where the circle on the major axis is intersected by the minor axis produced, a tangent is drawn to the ellipse; find the point of contact.

23. If from the extremities of the minor axis two straight lines be drawn through any point in the ellipse, and intersect the axis major in Q and R, then $CQ \, . \, CR = CA^2$.

24. If a rod slide between a vertical wall and a horizontal plane, any point on it traces out an ellipse.

25. If a tangent be drawn to the interior of two concentric ellipses, the axes of which are in the same straight line, meeting the exterior one in P, Q, and at P, Q tangents be drawn to the latter, intersecting in R, prove that the locus of R is an ellipse.

26. To shew that the locus of one end of a given straight line, whose other end, and a given point in it move in straight lines at right angles to one another, is an ellipse.

27. If with the co-ordinates of any point in an elliptic quadrant as semi-axes, a concentric ellipse be described, the chord of the quadrant of the one will be a tangent to the other.

28. The locus of the centre of a circle touching two circles externally is an hyperbola.

29. The locus of the centre of a circle touching one circle externally and one internally is an hyperbola.

30. Find the locus of the extremity of the perpendicular from the centre on the tangent to the hyperbola.

31. Find the polar equation to the hyperbola, focus pole, from the equation $SP - HP = 2a$.

32. If a tangent at any point of an hyperbola cut the tangents at the vertices in T and T', then $AT \cdot A'T' = b^2$.

33. If $3AC = 2CS$ in an hyperbola, find the inclination of the asymptotes to the transverse axis.

34. If from a point P in an hyperbola, PK be drawn parallel to the transverse axis cutting the asymptotes in I and K, then $PK.PI = a^2$, or if parallel to the conjugate, $PK \cdot PI = b^2$.

35. If from a point P in an hyperbola, PN be drawn parallel to the asymptote to meet the directrix in N, then $PN = SP$.

36. If A, A' be the extremities of the transverse axis of an hyperbola, T the point where the tangent at P meets AA', QTR a line perpendicular to AA' and meeting AP, $A'P$ in Q and R respectively, then $QT = TR$.

37. Find the eccentricity and latus rectum of the hyperbola

$$\{y^2 = 4(x^2 + a^2)\}.$$

38. In the equilateral hyperbola, the eccentricity is the ratio of the diagonal of a square to its sides.

39. A tangent at any point P of an ellipse meets the axis major produced in T, and the axis minor produced in t; to find the locus of a point Q in Tt, such that, $QT : Qt :: m : n$.

40. To find the locus of the intersection of the ordinate of any point in an ellipse produced, with the perpendicular from the centre upon the tangent at that point.

41. If the normal at P meet the axis major of an ellipse in G, and GK be drawn perpendicular to SP, $GK = e \,.\, PM$, where PM is the ordinate of P.

42. If SQ be drawn, always bisecting the angle PSC, in an ellipse, and equal to a mean proportional between SC and SP, find the eccentricity of the curve which is the locus of Q.

43. Two straight lines, such that the product of the tangents of their inclinations to the axis of x is constant, touch an ellipse; shew that the locus of their intersection is an ellipse, or hyperbola, according as the product is negative or positive.

44. Shew that the locus of the summit of a moveable right angle, one side of which touches one, and the other side the other of two confocal ellipses, is a concentric circle.

45. An ellipse and hyperbola have the same foci and coincident axes; they cut each other at right angles.

46. If P be any point in the hyperbola, S and H the foci, find the locus of the centre of the circle; (i) which is inscribed in SPH, (ii) which touches HP and SH, SP produced, and (iii) which touches SH and PS, PH produced.

47. If a tangent at any point of an hyperbola be intersected by the tangents at the vertices, in H and K, the circle on HK as diameter passes through the foci.

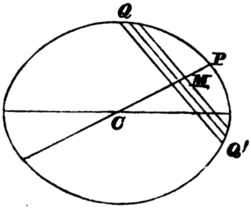
SECTION IX.

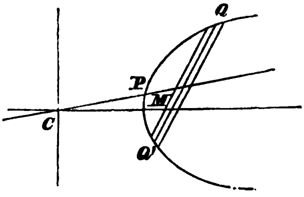
Central Conic Sections. Conjugate Diameters. The Asymptotes.

166. We saw (Art. 111), that there is only one system of rectangular axes about which the ellipse and hyperbola are so situated, that each axis should bisect all chords parallel to the other. We shall now shew that there are an infinite number of oblique axes which possess the above property with regard to the two curves.

167. *To find the locus of the middle points of any system of parallel chords.*

Let QQ' be one of the chords, M its middle point $(x'y')$, and





let the equation to QQ' be

$$\frac{y-y'}{s} = \frac{x-x'}{c} = l.$$

Now, if we substitute for x and y from this equation in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the result will refer to the points of intersection of the line and curve, and we shall have for the distance (l) of ($x'y'$) from those points

$$\frac{(cl + x')^2}{a^2} + \frac{(sl + y')^2}{b^2} = 1,$$

the roots of which equation (MQ and MQ') are equal and (Art. 27) of opposite signs; hence, the coefficient of $l = 0$, or

$$\frac{cx'}{a^2} + \frac{sy'}{b^2} = 0,$$

which is a relation between the co-ordinates of the middle point of the chord QQ'. But since the chords are parallel, s and c are the same for them all, and the same relation holds for the middle points of all; hence the equation to the locus required is

$$\frac{cx}{a^2} + \frac{sy}{b^2} = 0,$$

which represents a straight line (CM) through the centre.

If m be the tangent of the angle which the chords make with the axis of x, $m = \frac{s}{c}$, and the equation is

$$y = -\frac{b^2}{a^2m}x.$$

The equation for the hyperbola is, of course,

$$y = \frac{b^2}{a^2m}x.$$

Cor. If ($x'y'$) be the point P where CM meets the curve, we have

$$m = -\frac{b^2x'}{a^2y'},$$

but the right hand member (Art. 135), represents the tangent of the angle which the tangent at ($x'y'$) makes with the axis of x. Hence the tangent at the extremity of CP is parallel to the chords bisected by that line, as it evidently should be, since it may be considered as the limiting position of any chord QQ', as it moves parallel to itself up to P.

168. The straight line which bisects any system of parallel chords is called a *diameter* of the curve, and the chords are called the *ordinates* of that diameter. We have seen above (Art. 167), that if the equation to one of the chords be

$$y = mx + c,$$

the equation to the diameter of which those chords are ordinates is

$$y = -\frac{b^2}{a^2m}x.$$

We see then, that all diameters pass through the centre, and, conversely, since m may have any value, all lines passing through the centre are diameters. We shall see, hereafter, that the same is true in the case of the parabola, but, the centre of the parabola being infinitely distant, all its diameters will be, consequently, parallel*.

* We may remark here, that when we solved the general equation of the second degree for y (Art. 101), we obtained

$$y = -\frac{Bx + E \pm \sqrt{Px^2 + Qx + R}}{2A},$$

and that, hence, the line

$$y = \frac{Bx + E}{2A}$$

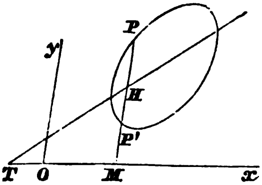
is a diameter to the curve, since for any given abscissa, as $OM\,(x')$ the differences of the ordinates of the line and curve are equal and of opposite sign; thus, in the figure,

$$PH = \frac{\sqrt{Px'^2 + Qx' + R}}{2A} = HP',$$

where HT is the line

$$y = -\frac{Bx + E}{2A},$$

and ∴ HT bisects all chords parallel to Oy.



169. *If two diameters be such, that one of them bisects all chords parallel to the other, then, conversely, the second will bisect all chords parallel to the first.*

Fig. 1.

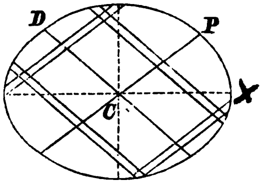
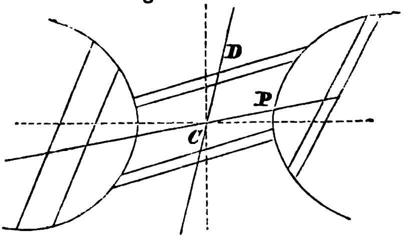


Fig. 2.



Let the diameter CP ($y = mx$) bisect all chords parallel to the diameter (CD) ($y = m'x$). Then, by Art. 168,

$$m = -\frac{b^2}{a^2m'};$$

hence

$$m' = -\frac{b^2}{a^2m},$$

or $y = m'x$ is the equation to the diameter which bisects all chords parallel to the diameter represented by $y = mx$.

170. Diameters so related that each bisects every chord parallel to the other, or, more commonly, such portions of these diameters as are intercepted by the curve, are called *Conjugate Diameters*, and the condition that the diameter ($y = mx$) should be conjugate to the diameter ($y = m'x$) is, by the last Article,

$$mm' = -\frac{b^2}{a^2}.$$

We shall see hereafter that only central curves can have conjugate diameters.

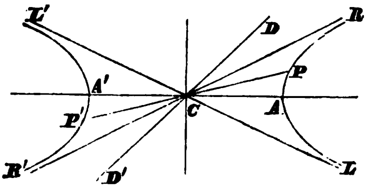
If θ, θ' be the angles which the conjugate diameters make with the major axis, we shall have

$$\text{in the ellipse, } \tan\theta \,.\, \tan\theta' = -\frac{b^2}{a^2},$$

$$\text{and in the hyperbola, } \tan\theta \,.\, \tan\theta' = \frac{b^2}{a^2};$$

hence, *in the ellipse the conjugate diameters fall on different sides of the axis minor;* for, if one of the angles θ, θ' be acute, and its tangent positive, the other must be obtuse and its tangent negative; and, by similar reasoning, we see that *in the hyperbola the conjugate diameters lie on the same side of the conjugate axis.*

171. Also, *of any two conjugate diameters, only one can meet the hyperbola;* for if one of the tangents ($\tan\theta$) be less than $\frac{b}{a}$, the other ($\tan\theta'$) must be greater than $\frac{b}{a}$, and it is evident from Art. 130, that the diameter which makes the angle θ with the axis will fall within the angle RCL, made by the asymptotes, and meet the curve, while the diameter that makes the angle θ' with the axis will fall within the angle RCL', and will not meet the curve, for it has been shewn that $\tan RCA = \frac{b}{a}$.



If in the hyperbola $\tan\theta = \tan\theta' = \pm\frac{b}{a}$, the two diameters coincide with the asymptote RR' or with $L'L$, according as we take the upper or lower sign; hence each asymptote is its own conjugate diameter.

If $\tan\theta = -\tan\theta' = \frac{b}{a}$, in the ellipse, the angles θ and θ' are supplementary, and by the symmetry of the figure the conjugate diameters are equal; hence, the equal conjugate diameters in an ellipse are parallel to chords joining the extremities of the major and minor axes, and if an ellipse and hyperbola have the same centre and axes, the equal conjugate diameters of the ellipse coincide with the asymptotes of the hyperbola.

172. When the ellipse becomes a circle, $b^2 = a^2$, and

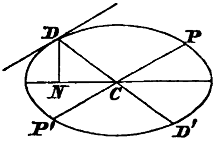
$$\tan\theta \, . \tan\theta' = -1,$$

or all conjugate diameters of the circle are at right angles to one another, and when the hyperbola is rectangular,

$$\tan\theta \, . \tan\theta' = 1,$$

$$\text{and } \theta + \theta' = \frac{\pi}{2} \text{ or } = \frac{3\pi}{2}.$$

173. We saw (Art. 151), that, when the curve is referred to its axes, as axes of co-ordinates, the polar of any point in the axis of x is parallel to the axis of y and *vice versâ*. This may be now seen to be a property of all conjugate diameters, including the axes as a particular case. For, if the equation to any diameter CP be $y = mx$, the equation to its conjugate CD is $y = -\frac{b^2}{a^2 m}x$. Now if (XY) be any point on CD, the equation to the polar of (XY) is



$$\frac{xX}{a^2} + \frac{yY}{b^2} = 1, \quad . \quad . \quad . \quad (1)$$

which is the equation to a line making with the axis of x an angle whose tangent $= -\frac{b^2 X}{a^2 Y}$; but this is $= m$, for

$$Y = -\frac{b^2}{a^2 m}X,$$

and therefore (1) is parallel to the line $(y = mx)$. Hence *the polar of any point in* CD *is parallel to* CP, ***and*** ***vice versâ.***

If (XY) be the point D where the diameter meets the curve, the equation to the polar

$$\frac{xX}{a^2}+\frac{yY}{b^2}=1$$

will represent the tangent at D; hence, *the tangent at the extremity of any diameter is parallel to its conjugate*, as we saw in Art. 167, Cor.

174. *The co-ordinates of the extremity of any diameter being given, to find those of the extremity of the diameter conjugate to it.*

Let CP, CD (fig. Art. 173) be a pair of conjugate diameters in the ellipse, and let x', y' be the co-ordinates of P; then the equation to CP is

$$y=\frac{y'}{x'}x, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and therefore the equation to CD (Art. 169), is

$$y=-\frac{b^2x'}{a^2y'}x. \quad . \quad . \quad . \quad . \quad (2)$$

To find the co-ordinates of D (xy) we have from (2) the relation

$$\frac{ay}{bx}=-\frac{bx'}{ay'}; \quad . \quad . \quad . \quad . \quad (3)$$

and also

$$a^2y^2+b^2x^2=a^2b^2=a^2y'^2+b^2x'^2, \quad . \quad . \quad . \quad . \quad (4)$$

since (xy), $(x'y')$ are points on the curve;

$$\text{hence from (3) } \frac{a^2y^2+b^2x^2}{b^2x^2}=\frac{a^2y'^2+b^2x'^2}{a^2y'^2},$$

$$\text{or from (4) } b^2x^2=a^2y'^2,$$

$$x=\pm\frac{a}{b}y';$$

$$\text{hence } CN = -\frac{a}{b}y', \text{ or } \frac{x}{a} = -\frac{y'}{b},$$

$$\text{and from (2) } DN = \frac{b}{a}x', \text{ or } \frac{y}{b} = \frac{x'}{a}.$$

The other pair of values $\left(x = \frac{a}{b}y',\ y = -\frac{b}{a}x'\right)$ have reference to the extremity D'.

175. In the hyperbola, if the diameter CP meet the curve, CD will not (Art. 170) meet it, and, indeed, it is evident that the method of the last Article will give us imaginary values of the co-ordinates of D; for as in equation (3) of the last Article, we have for the co-ordinates of D (xy),

$$\frac{ay}{bx} = \frac{bx'}{ay'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{hence } \frac{a^2y^2 - b^2x^2}{b^2x^2} = \frac{-(a^2y'^2 - b^2x'^2)}{a^2y'^2}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which would give $b^2x^2 = -a^2y'^2$; but, if we take D (xy) as the point where CD meets the conjugate hyperbola whose equation is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1,$$

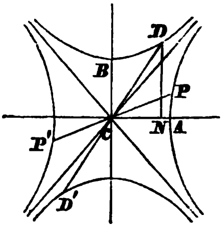
we have

$$a^2y^2 - b^2x^2 = a^2b^2 = -(a^2y'^2 - b^2x'^2),$$

whence, from (2) and (1)

$$x = CN = \frac{a}{b}y',$$

$$y = DN = \frac{b}{a}x'.$$



The other pair of values have reference to D'.

We shall therefore *define* the extremity of the diameter conjugate to CP, as '*the point where it meets the conjugate hyperbola.*'

It is evident, by exactly the same reasoning as we have used in the case of the hyperbola itself, that, if we consider CD as a diameter of the conjugate hyperbola, CP will be the diameter conjugate to CD; and whatever is proved of the point P, as a point in the hyperbola itself, is true of the point D, as a point in the conjugate hyperbola; for instance, (Art. 173), the tangent at D is parallel to CP, and so forth.

176. *In the ellipse, the* sum *of the squares of any two semi-conjugate diameters is equal to the sum of the squares of the semi-axes; and, in the hyperbola, the same is true of their* difference.

Let $P(x'y')$ and $D(x''y'')$ be the extremities of any two semi-conjugate diameters $CP(a')$, and $CD(b')$; then

$$a'^2 = CP^2 = x'^2 + y'^2,$$

$$\text{and } b'^2 = CD^2 = x''^2 + y''^2$$

$$= \frac{a^2y'^2}{b^2} + \frac{b^2x'^2}{a^2}. \quad \text{(Art. 174).}$$

$$\therefore\ a'^2 + b'^2 = (a^2 + b^2) \cdot \left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right)$$

$$= a^2 + b^2, \text{ in the ellipse;}$$

since, in that case,

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1.$$

$$\text{Also } a'^2 - b'^2 = (a^2 - b^2) \cdot \left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right)$$

$$= a^2 - b^2 \text{ in the hyperbola;}$$

since, in that case,

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1.$$

Cor. If $a^2 = b^2$, we have, in the hyperbola,

$$a'^2 - b'^2 = 0,$$

or every diameter in the equilateral hyperbola equals its conjugate.

177. *The rectangle contained by the focal distances of any point is equal to the square of the corresponding semi-conjugate diameter.*

In the ellipse $CD^2 = a^2 + b^2 - CP^2$, (Art. 176),

$$\text{but } CP^2 = x'^2 + y'^2 = x'^2 + b^2\left(1 - \frac{x'^2}{a^2}\right)$$

$$= b^2 + e^2x'^2;$$

$$\therefore\ CD^2 = a^2 - e^2x'^2 = (a - ex').(a + ex')$$

$$= HP.SP. \quad \text{(Art. 152).}$$

A similar investigation for the hyperbola would lead us to the same result.

178. *To find the length of the perpendicular from the centre on the tangent at any point* P (x′y′), *in terms of the semi-diameter, conjugate to* CP.

The length of a perpendicular from the origin, on the line

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

$$\text{is (Art. 42, Cor.)} = \frac{1}{\sqrt{\left(\dfrac{x'^2}{a^4} + \dfrac{y'^2}{b^4}\right)}}$$

$$= \frac{ab}{\sqrt{\left(\dfrac{b^2x'^2}{a^2} + \dfrac{a^2y'^2}{b^2}\right)}}$$

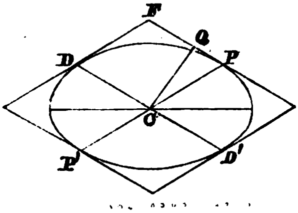
$$= \frac{ab}{b'},$$

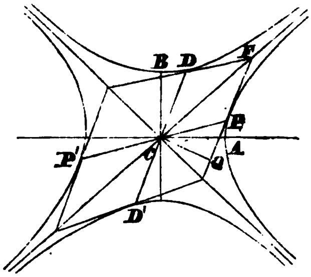
for $b'^2 = \frac{b^2x'^2}{a^2} + \frac{a^2y'^2}{b^2}$ by Arts. 174, 176.

179. *All parallelograms, whose sides are formed by lines passing through the extremity of one diameter and parallel to its conjugate, are equal in area.*

By Article 173, these lines are tangents to the curve, for it is there proved that the tangent at the extremity of any diameter is parallel to its conjugate; and, by Art. 136, the tangents at the extremities of any diameters are parallel to one another.

Let PP', DD' be two conjugate diameters, and let the sides of the parallelogram be tangents at P, P', D, D'. Then the area of the whole parallelogram





$$= 4 \text{ times the parallelogram } CPFD$$

$$= 4 \,.\, CQ \,.\, CD,$$

if CQ be the perpendicular from the centre on the tangent at P,

$$= 4\frac{ab}{b'} \,.\, b' \quad \text{(by Art. 178)}$$

$$= 4ab.$$

180. From the last Article may be found the angle between any two conjugate diameters which are given; for, if this angle be $= \gamma$, and the given semi-conjugate diameters be a', b', we have

$$a'b' \sin\gamma = \text{parallelogram } CPFD$$
$$= ab;$$

$$\text{hence, } \sin\gamma = \frac{ab}{a'b'}.$$

This equation together with the relations

$$a'^2 + b'^2 = a^2 + b^2, \text{ in the ellipse,}$$
$$a'^2 - b'^2 = a^2 - b^2, \text{ in the hyperbola,}$$

determine also the magnitude of two conjugate diameters that contain an angle γ.

181. The angle γ, PCD, in the ellipse, is always, except in the case of the axes, greater than a right angle, P being supposed in the first and D in the second quadrant; for if m and m' be the tangents of the angles that CP and CD make with the major axis,

$$\tan\gamma = \frac{m' - m}{1 + mm'}, \text{ ....} (1)$$

$$\text{and, since } m' = -\frac{b^2}{a^2 m}, \text{ we have,}$$

$$\tan\gamma = -\frac{(a^2m^2 + b^2)}{m(a^2 - b^2)}, \text{ ....} (2)$$

and therefore, since m is always positive, γ is $> 90^0$, unless $m = 0$, in which case $\gamma = 90^0$.

When $a = b$, $\tan\gamma = \infty$, or the conjugate diameters of the circle are all at right angles to each other, as we found in Arts. 94, 172.

Solving equation (2) as a quadratic in m, we have

$$m = \frac{-(a^2 - b^2)\tan\gamma \pm \sqrt{(a^2 - b^2)^2\tan^2\gamma - 4a^2b^2}}{2a^2} \quad \ldots\ldots\ldots (3)$$

Hence, the smallest negative value that $\tan\gamma$ can have, (since it is always negative,) is that which makes the quantity under the root in (3) vanish, which is

$$\tan\gamma = -\frac{2ab}{a^2 - b^2}.$$

This gives $m = \frac{b}{a}$, from (3), which is, therefore, the value of m which makes γ the largest; hence, (Art. 171) the equal conjugate diameters include the greatest angle.

182. *To find the equation to the ellipse or hyperbola, when referred to any two conjugate diameters* (2a′, 2b′) *as axes.*

We saw (Art. 111), that, of *rectangular* axes, there is only one system, to which, when a central curve is referred, its equation is of the form

$$Ax^2 + Cy^2 = F,$$

an equation which asserts that all chords parallel to one axis are bisected by the other. But every diameter (Art. 169) bisects the chords parallel to its conjugate; hence, there are an infinite number of *oblique* axes, which will give the equation in the above form, the only limitation being that they should pass through the centre, and that the angles (θ, θ') which they make with the major or transverse axis should be subject to the condition

$$\tan\theta \, . \tan\theta' = -\frac{b^2}{a^2}, \text{ in the ellipse,}$$

$$\tan\theta \, . \tan\theta' = \frac{b^2}{a^2}, \text{ in the hyperbola.}$$

Hence, the reasoning of Arts. 113, 114 applies to conjugate diameters, in precisely the same manner as to the axes of the curve, and the equations to the ellipse and hyperbola, referred to any pair of conjugate diameters, the parts of which intercepted by the curve are $2a'$ and $2b'$, are

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1,$$

$$\text{and } \frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1.$$

183. Since this equation is of precisely the same form as the equation to the curve referred to its axes, it follows that every property that has been deduced from the latter may be deduced from the former, so long as those properties do not depend upon the inclination of the axes, and, with this limitation, everything which has been proved of the axes, is true of any pair of conjugate diameters.

For example, the equation to the tangent at any point $(x'y')$, when the curve is referred to any conjugate diameters $(2a', 2b')$, is

$$\frac{xx'}{a'^2} + \frac{yy'}{b'^2} = 1,$$

and the intercept on the axis of x, and the subtangent, are, as before, $\frac{a'^2}{x'}$ and $\frac{a'^2 - x'^2}{x'}$. Also, if we wished to draw a tangent to the curve from an external point $(x'y')$, we should have, as in Article 147,

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1,$$

$$\frac{xx'}{a'^2} + \frac{yy'}{b'^2} = 1,$$

as equations to determine the points of contact.

184. We may obtain a simple geometrical method for constructing a pair of conjugate diameters, containing a given angle, as follows.

It was proved (Arts. 166—169) that, when the curve is referred to its axes, the condition that the two lines

$$y = mx, \quad y = m'x$$

should represent two conjugate diameters is

$$mm' = -\frac{b^2}{a^2};$$

and, as the reasoning of those Articles is equally applicable when the curve is referred to any pair of conjugate diameters*, we see that when the curve is referred to a pair whose semi-lengths are a', b', the condition is

$$mm' = -\frac{b'^2}{a'^2},$$

where, however, m and m' do not now as in (Art. 170) represent $\tan\theta$ and $\tan\theta'$, but

$$m = \frac{\sin\theta}{\sin(\omega-\theta)}, \quad m' = \frac{\sin\theta'}{\sin(\omega-\theta')},$$

ω being the angle between the axes.

Now, if the curve be referred to these two diameters, the equation may be written

$$y^2 = -\frac{b'^2}{a'^2}(x^2 - a'^2),$$

---

[*] In finding the locus of the middle points of any system of parallel chords (Art. 167), we must, in the case of *any* conjugate diameters, modify the proof, by assuming the equation to any one of the chords to be (Art. 27)

$$\frac{y-y'}{s} = \frac{x-x'}{c} = l,$$

where s now $= \frac{\sin\alpha}{\sin\omega}$, $c = \frac{\sin(\omega-\alpha)}{\sin\omega}$.

which equation may be split into the two

$$y = -k\frac{b'^2}{a'^2}(x - a'),$$

$$y = \frac{1}{k}(x + a'),$$

where k is a perfectly arbitrary quantity; now these two lines

(i) pass through the extremities of the diameter which is the axis of x,

(ii) intersect in the curve, since by eliminating k between them we have the equation to the curve,

(iii) fulfil the same conditions, with regard to their inclinations to the axis of x as a pair of conjugate diameters, for

$$\left(\frac{1}{k}\right) \cdot \left(-k\frac{b'^2}{a'^2}\right) = -\frac{b'^2}{a'^2}.$$

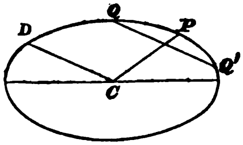
Lines drawn in this manner from the extremities of any diameter, to a point in the curve, are called *Supplemental Chords*, and it is evident that diameters parallel to any pair of supplemental chords are conjugate.

Hence, to draw a pair of conjugate diameters containing any given angle, describe on any diameter a segment of a circle containing that angle, and join the points where it meets the curve with the extremities of the assumed diameter. We thus obtain a pair of supplemental chords inclined at the given angle. The lines drawn through the centre parallel to these will be the required conjugate diameters.

The property of supplemental chords was demonstrated for the circle (Art. 79), when it was shewn that all supplemental chords in the circle are at right angles, as are all conjugate diameters.

185. ***Tangents at the extremities of any chord intersect in the diameter of which the chord is an ordinate.***

Taking that diameter and its conjugate as the axes of x and y, the equation to the tangent will be



$$\frac{xx'}{a'^2} \pm \frac{yy'}{b'^2} = 1,$$

according as we take Q $(x'y')$ or Q' $(x', -y')$ as the point of contact.

In each case, when $y = 0$, x has the same value $= \frac{a'^2}{x'}$, or the tangents meet CP produced in the same point.

This follows from Art. 167; for, if the equation to a chord be $y = mx + b'$, and $(x'y')$ the point of intersection of tangents at its extremities, we have also for the equation to the chord (Art. 147)

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

a line which makes with the axis of x an angle whose tangent $= -\frac{b^2x'}{a^2y'}$. Hence $m = -\frac{b^2x'}{a^2y'}$, or $y' = -\frac{b^2}{a^2m}x'$, which is the equation to the diameter, of which $y = mx + b'$ is an ordinate; hence $(x'y')$ is a point in that diameter.

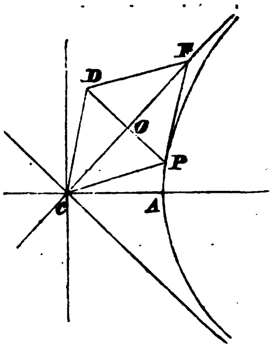
THE ASYMPTOTES.

186. The following Articles relate exclusively to the hyperbola, as the ellipse is limited in every direction, and, therefore, cannot have an asymptote.

The diagonals of all parallelograms, described as in Art. 179, are parallel to the asymptotes.

Join PD, then the co-ordinates of O, the middle point of PD, are (Art. 36),

$$x = \tfrac{1}{2}\left(x' + \frac{ay'}{b}\right), \quad y = \tfrac{1}{2}\left(y' + \frac{bx'}{a}\right);$$



since the co-ordinates of P are x', y', and of D, $\frac{ay'}{b}$, $\frac{bx'}{a}$ (Art. 174). Now CF passes through O, hence

$$\tan FCA = \frac{y' + \frac{bx'}{a}}{x' + \frac{ay'}{b}} = \frac{b}{a},$$

which is the tangent of the angle which the asymptote makes with the transverse axis (Art. 130); hence CF coincides with one of the asymptotes.

Also, since the diagonal PD passes through the points P $(x'y')$ and D $\left(\frac{ay'}{b}\ \frac{bx'}{a}\right)$, the tangent of the angle it makes with the transverse axis is (Art. 28)

$$= \frac{y' - \frac{bx'}{a}}{x' - \frac{ay'}{b}} = -\frac{b}{a},$$

which is the tangent of the angle which the other asympt makes with the same axis; hence PD is parallel to the oth asymptote.

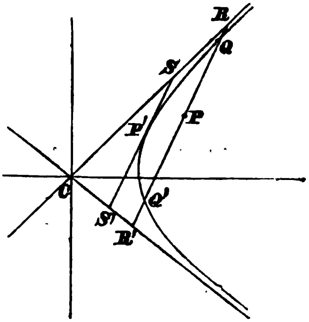
Cor. It may be seen from this Art. that the for of the equation to the asymptotes remains the same when t curve is referred to any conjugate diameters; for, if CP (and CD (b') be taken as axes, the co-ordinates of O are $\frac{1}{2}$ $\frac{1}{2}b'$, and therefore the equation to CF is (Art. 23, Cor.)

$$y = \frac{b'}{a'}x,$$

and similarly for the other asymptote.

187. Hence, if any two conjugate diameters CP, CD be given in position, we can find the asymptotes, by completing the parallelogram $CPFD$, the diagonals of which will shew the direction of the asymptotes; or, if the asymptotes be given, we can find the position of the diameter conjugate to any diameter CP, whose position is given; for, if we draw PO parallel to one asymptote to meet the other in O, and produce PO to D, making $OD = PO$, CD will be the diameter conjugate to CP.

188. *If any line* (RR′) *cut an hyperbola and its asymptotes, the portions* (RQ), (R′Q′), *intercepted between the curve and the asymptotes are equal.*



Since the equations to the two asymptotes are

$$\frac{x}{a} - \frac{y}{b} = 0,$$

$$\frac{x}{a} + \frac{y}{b} = 0,$$

the equation to the two, considered as one locus, is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \quad \ldots \ldots \quad (1)$$

and the equation to the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad \ldots \ldots \quad (2)$$

Now let the equation to RR', passing through a point P $(x'y')$ be

$$\frac{y - y'}{s} = \frac{x - x'}{c} = l; \quad \ldots \ldots \quad (3)$$

then, for the distances of P from R and R', we have from (1) and (3)

$$\frac{(cl+x')^2}{a^2} - \frac{(sl+y')^2}{b^2} = 0, \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

and for the distances of P from Q and Q', from (2) and (3),

$$\frac{(cl+x')^2}{a^2} - \frac{(sl+y')^2}{b^2} = 1. \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

If P be the middle point of RR', the roots of equation (4) are equal and of opposite signs, and if P be the middle point of QQ' the roots of (5) are equal and of opposite signs. But these two suppositions evidently give the same condition, viz.

$$\frac{cx'}{a^2} - \frac{sy'}{b^2} = 0, \quad . \quad . \quad . \qquad (6)$$

hence, if P be the middle point of RR' it is also the middle point of QQ', and we have

$$PQ = PQ', \; PR = PR', \text{ and } \therefore \; RQ = R'Q'.$$

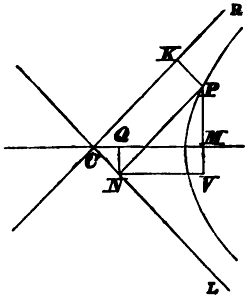
Cor. Equation (6), which is the condition that $(x'y')$ should be the middle point of the chord RR', is also (Art. 135) the condition that the line should be a tangent to the curve at the point $(x'y')$; hence if the middle point $(x'y')$ of RR' l on the curve, RR' is a tangent, or *the portion of the tange intercepted by the asymptotes is bisected at the point of co tact.* This follows directly from the consideration that th tangent $SP'S'$ is the limiting position of RR' when it is mov parallel to itself up to P'.

189. *From the equation to the hyperbola referred to i axes, to derive the equation when referred to its asymptot as axes.*

Let the lower asymptote be taken for axis of x, and l

CM, *PM* be the original co-ordinates, *CN*, *PN* the new ones; draw *NQ* perpendicular to *CM* and *NV* parallel to it, meeting *PM* produced in *V*, and let angles *RCM*, *LCM* each $= \alpha$; then $\tan \alpha = \frac{b}{a}$, and, if $a^2 + b^2 = m^2$, we have

$$\cos \alpha = \frac{a}{m}, \ \sin \alpha = \frac{b}{m}.$$



$$\text{Now } CM = NV + CQ = PN \cos \alpha + CN \cos \alpha,$$

$$PM = PV - QN = PN \sin \alpha - CN \sin \alpha;$$

hence we must write, in the equation to the hyperbola,

$$\text{for } x, \ (y + x) \cos \alpha \text{ or } \frac{a(y + x)}{m},$$

$$\text{for } y, \ (y - x) \sin \alpha \text{ or } \frac{b(y - x)}{m},$$

$$\text{which gives } (y + x)^2 - (y - x)^2 = m^2,$$

$$\text{or } xy = \frac{m^2}{4},$$

which is therefore the required equation.

190. The result of the last Article may be obtained independently as follows.

Let *P* be any point $(x' y')$ in the hyperbola, when referred to its axes; then the equation to the asymptote *CR*, which makes an angle α with the axis is

$$ay - bx = 0, \quad . \quad . \quad . \quad . \quad (1)$$

and to *CL*, which makes an angle $\pi - \alpha$ with the axis,

$$ay + bx = 0, \quad . \quad . \quad . \quad . \quad (2)$$

and the equation to PK, drawn parallel to CL is

$$\frac{y-y'}{s}=\frac{x-x'}{c}=l; \quad . \quad . \quad . \quad . \quad (3)$$

where $s=\sin(\pi-\alpha)=\frac{b}{m}$, $c=\cos(\pi-\alpha)=-\frac{a}{m}$.

From (1) and (3) we have

$$a(sl+y')-b(cl+x')=0,$$

which gives

$$l \text{ or } PK=\frac{bx'-ay'}{as-bc}$$

$$=\frac{m(bx'-ay')}{2ab}.$$

In the same manner, if PN be parallel to CR, we have

$$PN=\frac{m(bx'+ay')}{2ab}.$$

$$\text{Hence } PK.PN=\frac{m^2(b^2x'^2-a^2y'^2)}{4a^2b^2},$$

$$\text{or } xy=\frac{m^2}{4},$$

$$\text{since } b^2x'^2-a^2y'^2=a^2b^2.$$

191. *To find the equation to the tangent at any poi* (x′y′) *when the asymptotes are the co-ordinate axes.*

Let ω be the angle between the asymptotes, and let t equation to a line cutting the hyperbola in $(x'y')$ be (Art. 27)

$$\frac{y-y'}{s}=\frac{x-x'}{c}=l, \quad . \quad . \quad (1)$$

$$\text{where } s=\frac{\sin\alpha}{\sin\omega}, \quad c=\frac{\sin(\omega-\alpha)}{\sin\omega};$$

then, substituting from this equation in the equation to the hyperbola,

$$xy = \frac{m^2}{4}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

we have

$$(sl + y')(cl + x') = \frac{m^2}{4},$$

$$\text{or } scl^2 + (sx' + cy')\,l = 0, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

since $(x'y')$ is a point on the curve, and therefore

$$x'y' = \frac{m^2}{4}.$$

Equation (3) will give us, by reasoning exactly similar to that used in Arts. 82, 135, as a condition that equation (1) should represent a tangent at $(x'y')$,

$$sx' + cy' = 0, \quad . \quad . \quad . \quad . \quad (4)$$

$$\text{and from (1) } \frac{y - y'}{x - x'} = \frac{s}{c},$$

$$= -\frac{y'}{x'} \text{ from (4)}$$

$$\text{whence } yx' + xy' = 2x'y',$$

$$\text{or } yx' + xy' = \frac{m^2}{2},$$

which is the required equation.

192. The equation to the polar of any point $(x'y')$, when the asymptotes are co-ordinate axes, will be readily found, as in Art. 148, to be

$$x'y + y'x = \frac{a^2 + b^2}{2}.$$

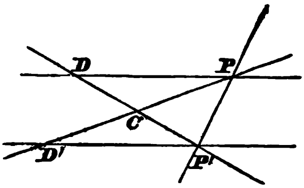
We can make use of this equation to shew on which branch of the hyperbola the tangents fall that are drawn from any external point $(x'y')$. The points of contact are determined by the intersection of the line

$$x'y + y'x = \frac{a^2 + b^2}{2},$$

with the curve (Art. 147); now the intercepts of this line on the asymptotes, obtained by putting $y = 0$ and then $x = 0$ in the equation, are

$$x = \frac{a^2 + b^2}{2y'},\ y = \frac{a^2 + b^2}{2x'};$$

and it is easily seen that, if the intercepts of any line as PP' on the asymptotes have the *same* sign, the line will cut that branch of the hyperbola which is within the angle PCP' in two points; but, if the intercepts of any line as DP have *different* signs, the line will cut each branch of the curve in one point. In the present case, the signs of the intercepts wi depend upon the signs of x' and y'; hence, if x' and y' be bot positive or both negative, that is, if the point $(x'y')$ be withi the angles PCP' or DCD', the tangents will fall upon tha branch of the curve which lies in the same angle with the poin $(x'y')$. If x' be negative and y' positive, or the point $x'y'$ li in the angle PCD, the polar will lie as the line $P'D'$, and th tangents fall on both branches of the curve; and similarly, i the point lies in the angle $P'CD'$ the polar will lie as the lin PD, and the tangents fall upon both branches of the curve.



EXAMPLES VIII.

1. If θ be the angle between the tangent and the focal distance at any point of an ellipse, the distance of that point from the centre

$$=(a^2-b^2\cot^2\theta)^{\frac{1}{2}}.$$

2. If CP, CQ be semi-diameters, at right angles to each other,

$$\frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{a^2}+\frac{1}{b^2}.$$

3. If from the focus S of an ellipse perpendiculars be drawn on CP, CD, conjugate diameters, these perpendiculars produced backwards will intersect CD and CP in the directrix.

4. If ρ, r and ρ', r' be respectively the focal distances of two points, P, D, the extremities of a pair of conjugate diameters of an ellipse, then

$$\rho r+\rho' r'=a^2+b^2.$$

5. If a tangent to an hyperbola at P cut off CT, Ct from the axes, then, $PT\,.\,Pt=CD^2$, CD being the semi-conjugate diameter.

6. In the equilateral hyperbola the conjugate diameters make equal angles with the asymptotes.

7. From the extremities P, D of two conjugate diameters, normals are drawn to the major axis of an ellipse; the sum of the squares of these two $=\frac{b^2}{a^2}(a^2+b^2)$.

8. If the tangent at the vertex A cut any two conjugate diameters of an ellipse produced in T and t, then, $AT\,.\,At=b^2$.

9. The diameters which bisect the lines joining the extremities of the axes of an ellipse, are equal and conjugate.

10. The locus of the middle points of chords of an ellipse which pass through a fixed point is a similar ellipse, and if the fixed point be the focus, the major axis of the ellipse is SC.

11. The tangent at any point of an hyperbola is produced to meet the asymptotes; shew that the triangle cut off is of constant area.

12. If, in two concentric and similarly situated hyperbolas, two points be taken, whose abscissæ are as their transverse axes, the locus of the middle point of the line joining them is an hyperbola, concentric and similarly situated, whose axes are arithmetic means between those of the given hyperbolas.

13. If any two tangents be drawn to an hyperbola, and their intersections with the asymptotes be joined, the joining lines will be parallel.

14. Shew that the locus of the points of quadrisection of all parallel chords in a circle is a concentric ellipse.

15. CP and CD are any conjugate diameters of an ellipse; join PD, draw CP' parallel to DP, and join PP'; then the area of the trapezium $CP'PD$ is to that of the ellipse as $1 + \frac{1}{\sqrt{2}} : 2\pi$.

16. If a length $PQ = CD$ be taken in the normal to an ellipse, the locus of the point Q is a circle whose radius $= a - b$ or $a + b$, according as Q is taken within or without the ellipse.

17. CP, CD are semi-conjugate diameters of an ellipse, and PF is a perpendicular let fall from P on CD or CD produced; determine the locus of F.

18. The chords joining the extremities of the conjugate diameters of an ellipse will all touch in their middle points an ellipse with axes $a\sqrt{2}$, $b\sqrt{2}$, parallel to those of the original curve.

19. If a circle be described from the focus of an hyperbola, with radius equal to half the conjugate axis, it will touch the asymptotes in the points where they are cut by the directrix.

20. If A and S be the vertex and focus of an hyperbola, and the tangent at the vertex and the directrix meet the asymptote in G and R respectively, then SG is parallel to AR.

21. The radius of a circle, which touches an hyperbola and its asymptotes, is equal to that part of the latus rectum produced which is intercepted between the curve and the asymptote.

22. In the hyperbola, if the perpendicular from the focus meet the asymptote in B, then $CB = CA$.

23. If S and H be the foci of an equilateral hyperbola, and a circle be described on SH, the tangents at the vertices will intersect the asymptotes in the circumference.

24. In an equilateral hyperbola, the distance of any point from the centre is a geometric mean between its distances from the foci.

25. In the equilateral hyperbola, CP equals the length of the normal at P.

26. If the tangents at the extremities of any diameter DD' of an ellipse be intersected by the tangent at any other point in T, T', then $DT \, . \, D'T' = CP^2$.

27. To find the locus of the intersection of tangents to an ellipse, which are parallel to conjugate diameters.

28. Find the equation to the locus of the middle points of all chords of a given length, in an ellipse.

29. If two concentric equilateral hyperbolas be described, the axes of the one being the asymptotes of the other, they will intersect at right angles.

30. If P be the middle point of a line AB, which is so drawn as to cut off a constant area from the corner of a square, its locus is an equilateral hyperbola.

31. If S and H be the foci of an equilateral hyperbola, and a circle be described upon SH, then the quadrantal chord of this circle shall be a tangent to that described upon the transverse axis.

32. If from a point in an equilateral hyperbola lines be drawn to the extremities of a diameter, these lines will make equal angles with the asymptotes.

33. If AA' be any diameter of a circle, PQ any ordinate to it, then the locus of the intersections of AP, $A'Q$ is an equilateral hyperbola.

34. In an equilateral hyperbola, focal chords parallel to conjugate diameters are equal.

35. If a series of lines have their extremities in two straight lines at right angles to one another, and all pass through a given point, the locus of their middle points is an equilateral hyperbola.

36. PQ is an ordinate to the axis major AA' of an ellipse, meeting the curve in P and Q; draw AP, $A'Q$ intersecting in R; the locus of R is an hyperbola with the same centre and axes.

37. If tangents be drawn, making a given angle with the axes of all ellipses having the same foci, the locus of the point of contact is an equilateral hyperbola.

38. If normals be drawn to an ellipse from a given point within it, the points where they meet the curve will all lie in an equilateral hyperbola which passes through the given point, and has its asymptotes parallel to the axes of the ellipse.

39. Find the locus of the middle points of chords in a circle which touch a concentric ellipse.

40. If normals be drawn from the extremities of conjugate diameters to an hyperbola, and the point of their intersection be joined to the centre, this line produced shall be perpendicular to the line passing through the extremities of the conjugate diameters.

41. Given in position, a right line AB and a point P outside it; a right line PM is drawn, intersecting AB in C, from the extremity M of which a perpendicular MD on AB intercepts CD of a given magnitude; find the locus of M.

42. The locus of the centres of all circles, which cut off from the directions of two sides of a triangle chords equal to two given straight lines, is an equilateral hyperbola, having two conjugate diameters in the directions of these sides.

43. A straight line passes through a given point and is terminated in the sides of a given angle; find the locus of the point which divides it in a given ratio.

44. From a point P perpendiculars are dropped upon the sides of a given angle, so as to contain a quadrilateral of given area; shew that the locus of P is an hyperbola whose centre is the vertex of the given angle.

45. Given the base of a triangle and the difference of the tangents of the base angles; shew that the locus of the vertex is an hyperbola, of which the perpendicular through the centre of the base is an asymptote.

46. If about the exterior focus of an hyperbola, a circle be described with radius equal to the semi-conjugate axis, and tangents be drawn to it from any point in the hyperbola, the line joining the points of contact will touch the circle described on the transverse axis as diameter.

47. If from the centre of an equilateral hyperbola a line be drawn through any point P, and if ϕ and ϕ' be the angles which this line and the polar of P respectively make with the transverse axis, then

$$\tan\phi \,.\, \tan\phi' = 1.$$

48. Find the locus of the middle points of a system of parallel chords drawn between an hyperbola and the conjugate hyperbola.

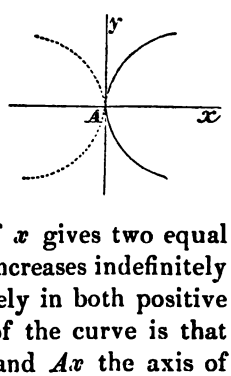
SECTION X.

The Parabola.

193. We saw (Art. 116) that there is one pair of rectangular axes, to which when the parabola is referred, its equation may be written in the simple form

$$y^2 = Lx,$$

and we shall now proceed to determine its form and principal properties from this equation.



Since $y = \pm\sqrt{Lx}$, when $x = 0$, $y = \pm 0$, or the origin is a point in the curve, and the line $(x = 0)$ meets the curve in two coincident points, that is to say, the axis of y is a tangent to the curve. No part of the curve can lie on the left side of the origin, for negative values of x would render y imaginary. It must be symmetrical with regard to the axis of x, since every value of x gives two equal values of y with opposite signs; also as x increases indefinitely in a positive direction, y increases indefinitely in both positive and negative directions; hence, the form of the curve is that of the figure: the point A is the vertex, and Ax the axis of the parabola.

If the equation be

$$y^2 = -Lx,$$

no positive values of x will give real values for y, but x may have any negative value; hence, in this case, no part of the curve lies on the right of the origin, and, by exactly the same reasoning as before, the curve may be seen to correspond to the dotted line in the figure. It may be proved to be concave to the axis of x by the method of Art. 142, but this is sufficiently evident from Art. 196.

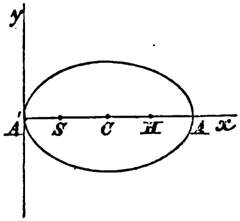
194. The parabola, like the hyperbola, has infinite branches, with this important difference in their nature. The tangent to the hyperbola, and consequently the direction of the branch, tends ultimately to coincide with a line making a finite angle with the axis of x, viz. the asymptote; while the tangent to the parabola, as will be shewn hereafter (Art. 199, 200), tends ultimately to become parallel to that axis and infinitely distant from it.

195. We see then that the parabola is, as it was defined (Art. 101), a curve limited in one direction, and unlimited in the opposite direction. We saw also (Art. 109), that the parabola might be considered as a central curve, with its centre removed to an infinite distance. We may therefore regard the parabola as an elongated ellipse; and as this analogy is very useful in enabling us to foresee the properties of the curve, we shall prove the following proposition.

196. *If we suppose the distance between one vertex and focus of an ellipse to be given, while the axis major increases without limit, the curve will ultimately become a parabola.*

The equation to the ellipse, when the vertex A' is origin, is, (Art. 126),

$$y^2 = \frac{b^2}{a^2}(2ax - x^2),$$



and, in order to find out what this equation becomes under the proposed circumstances, it will be necessary to express b in terms of a and the distance (d) $A'S$

between the vertex and focus which is supposed to remain finite. Now,

$$A'S + SC = a,$$

$$\text{or } d + \sqrt{a^2 - b^2} = a,$$

$$\text{whence } b^2 = 2ad - d^2;$$

and the equation becomes

$$y^2 = \frac{2ad - d^2}{a^2}(2ax - x^2),$$

$$\text{or } y^2 = \left(2d - \frac{d^2}{a}\right) \cdot \left(2x - \frac{x^2}{a}\right),$$

or, when $a = \infty$,

$$y^2 = 4dx,$$

which is the equation to a parabola.

Since $\frac{b^2}{a^2} = \frac{2ad - d^2}{a^2}$, and therefore vanishes when a is infinite, we have $e^2 = 1 - \frac{b^2}{a^2} = 1$, when $a = \infty$; hence, the parabola may be considered as an ellipse whose eccentricity $= 1$.

The same property may be proved in the same manner for the hyperbola: in that case, $b^2 = 2ad + d^2$, which value must be substituted in the equation

$$y^2 = \frac{b^2}{a^2}(2ax + x^2).$$

197. We shall, for the future, use the equation to the parabola, in the form

$$y^2 = 4dx,$$

derived from its analogy with the ellipse, and we shall call that point on the axis of x, at a distance $= d$ from the vertex, which was the focus of the ellipse, the *focus* of the parabola.

use this in drawing curve.

198. *To find the equation to a straight line touching the parabola in the point* (x′y′).

We shall proceed exactly as in the case of the circle and ellipse, and shall simply point out the steps in the proof.

Let the equation to a line cutting the curve in $(x'y')$, be

$$\frac{y-y'}{s} = \frac{x-x'}{c} = l; \dots\dots\dots\dots\dots (1)$$

then, for the distances of $(x'y')$ from the points of section of the line and the parabola $(y^2 = 4dx)$, we have

$$(sl + y')^2 = 4d(cl + x'),$$

$$\text{or } s^2l^2 + 2(sy' - 2dc)\,l = 0 \dots\dots\dots\dots (2)$$

$$\text{Since } y'^2 = 4dx'.$$

Equation (2) gives us

$$l = 0,$$

as it should, and

$$s^2l + 2(sy' - 2dc) = 0.$$

If the line (1) be a tangent at $(x'y')$, l vanishes, and we have

$$sy' - 2dc = 0.$$

Eliminating s and c by this equation and equation (1), we have

$$(y - y')\,y' = 2d(x - x');$$

$$\text{or } yy' = 2dx - 2dx' + y'^2,$$

or, since $y'^2 = 4dx'$, we have

$$yy' = 2d(x + x'),$$

for the equation to the tangent at $(x'y')$.

If m be the trigonometrical tangent of the angle which the tangent at $(x'y')$ makes with the axis of x, we have

$$m = \frac{s}{c} = \frac{2d}{y'}.$$

199. If α be the angle which the tangent makes with the axis of x, we have

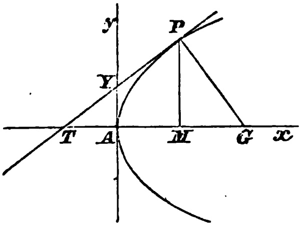
$$\tan \alpha = \frac{2d}{y'}.$$

When $y' = 0$, $\tan \alpha = \infty$, or the tangent at the vertex is perpendicular to the axis, as we saw in Art. 193. Also as y' increases from 0 to ∞, $\tan \alpha$ decreases from ∞ to 0, or the tangent tends continually to become parallel to the axis.

200. If we make $y = 0$ in the equation to the tangent, we have $x = -x'$ or $AT = AM$. Hence, the subtangent $MT = 2x'$, and is bisected at the vertex.

Also, writing $x = 0$ in the equation to the tangent, we have, for the intercept AY,

$$y = \frac{2dx'}{y'} = \frac{y'}{2};$$



hence, when x' and y' become infinite, the intercepts of the tangent at $(x'y')$ on the axes become infinite, or the parabola has no asymptotes, since an asymptote is the limiting position of the tangent, when the point of contact is removed to an infinite distance.

The property $AT = AM$ enables us to draw a tangent to a parabola at a given point in the curve; for, supposing AM given, we have only to take $AT = AM$ and join TP, which will be the tangent at P.

201. *To find the equation to the tangent in terms of its inclination to the axis.*

Proceeding as in Art. 198, we have for the equation to the tangent

$$\frac{y - y'}{s} = \frac{x - x'}{c}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

with the condition,

$$sy' - 2dc = 0, \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where we must now eliminate x' and y' instead of s and c.

From (2) and the equation

$$y'^2 = 4dx',$$

we have

$$y' = \frac{2dc}{s}, \qquad x' = \frac{dc^2}{s^2},$$

and from (1)

$$cy - sx = cy' - sx',$$

$$\therefore\ cy - sx = \frac{dc^2}{s},$$

the equation required.

If m be the tangent of the angle made by (1) with the axis of x, $m = \frac{s}{c}$, and the equation becomes

$$y = mx + \frac{d}{m}.$$

This equation may be obtained directly, without introducing the point $(x'y')$, as in the case of the circle, (Art. 86).

202. *To determine the tangents to a parabola which pass through a given point* (x'y').

The equation to the tangent is

$$y = mx + \frac{d}{m}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and, since it passes through $(x'y')$, we have

$$y' = mx' + \frac{d}{m};$$

$$\text{or } m^2 - \frac{y'}{x'}m + \frac{d}{x'} = 0\,; \quad . \quad . \quad (2)$$

which equation gives two values of m, and therefore shews that, in general, two tangents may be drawn to the curve through a given point.

If m, m' be the two roots of the equation, the equations to the two tangents will be

$$y = mx + \frac{d}{m}, \qquad y = m'x + \frac{d}{m'}.$$

203. *To determine the locus of the intersection of two tangents at right angles to one another.*

If in the last Article the two tangents are at right angles,

$$mm' = -1.$$

Hence, from (2), since mm' is the product of the roots,

$$\frac{d}{x'} = mm' = -1,$$

or $x = -d$ is the equation required, which represents a line perpendicular to the axis of x, at a distance $= d$ on the negative side of the origin. It will be seen hereafter (Art. 209) that this is the *directrix* of the parabola.

204. It will be remembered that the equation to the same locus in the case of the ellipse (Art. 143), was found to be a circle whose equation is

$$x^2 + y^2 = a^2 + b^2, \quad . \quad . \quad . \quad . \quad (1)$$

and we shall now shew that, when the ellipse passes into a parabola, this circle becomes the directrix. For transferring the origin in (1) to A', by writing $x - a$ for x, we have

$$x^2 - 2ax + y^2 - b^2 = 0\,; \quad . \quad . \quad . \quad (2)$$

but if $A'S = d$, $b^2 = 2ad - d^2$ (Art. 196), and (2) becomes

$$x^2 - 2ax + y^2 - 2ad + d^2 = 0. \quad . \quad . \quad (3)$$

Dividing (3) by a and then making a infinite, we obtain the equation

$$x = -d,$$

the equation (Art. 209) to the directrix of the parabola.

205. *To find the equation to the normal to a parabola at any point* (x′y′).

Since the normal passes through $(x'y')$, its equation is

$$y - y' = m(x - x'),$$

and, since it is perpendicular to the line

$$y - y' = \frac{2d}{y'}(x - x'),$$

we have $m = -\frac{y'}{2d}$, and the equation is therefore

$$y - y' = -\frac{y'}{2d}(x - x').$$

Putting $y = 0$ in this equation, we have, (fig. Art. 200), for the intercept of the normal on the axis of x,

$$x = 2d + x', \text{ or } AG = 2d + AM.$$

Also $MG = AG - AM$, or the *subnormal* $= 2d$.

206. *To find the equation to the chord joining the points of contact of two tangents drawn from any point* (x′y′).

The reasoning is precisely similar to that used in Arts. 89, 147; for, if $(x''y'')$ be one of the points of contact, the equation to one tangent is

$$yy'' = 2d(x + x''),$$

and, since it passes through $(x'y')$, we have

$$y'y'' = 2d(x' + x'');$$

hence, $(x''y'')$ is a point on the line represented by

$$y'y = 2d(x' + x);$$

and, for the same reason, the other point of contact is a point on this line, hence,

$$yy' = 2d(x + x')$$

is the equation to the chord joining the points of contact.

Hence, to draw tangents to the curve from any point $(x'y')$, we have the two equations

$$yy' = 2d(x + x'), \qquad y^2 = 4dx,$$

to determine the co-ordinates of the points of contact. These equations will generally give two points corresponding to the points of intersection of the line and curve.

207. *To find the equation to the polar of any point* (x′y′).

208. *If any point* (x″y″) *be taken on the polar of* (x′y′), *the polar of* (x″y″) *must pass through the point* (x′y′).

We leave these two problems for the student to work out, as the reasoning in no way differs from that used in the cases of the circle and central curves, (Arts. 90, 93, 148, 150).

The equation to the polar of $(x'y')$ will be found to be

$$yy' = 2d(x + x'),$$

and we may remark of this equation, as in the case of the curves that we have before considered, that it represents

(i) the locus of the intersection of tangents, drawn at the extremities of chords which pass through $(x'y')$, *whatever* be the position of that point;

(ii) the chord joining the points of contact of a pair of tangents, drawn to the curve from $(x'y')$, if $(x'y')$ be *without* the curve;

(iii) the tangent at $(x'y')$, if the point $(x'y')$ be *on* the curve.

209. If $y' = 0$, the equation to the polar becomes

$$x + x' = 0,$$

which shews us that the polar of any point on the axis of x is parallel to the axis of y.

The polar of the focus, whose co-ordinates are (Art. 197) $x = d$, $y = 0$, will have for its equation

$$x + d = 0,$$

a straight line perpendicular to the axis of the parabola, lying to the left of the vertex, and at a distance from it $= d$. The polar of the focus is called the directrix of the parabola.

210. *To find the distance of any point in the parabola from the focus.*

Since the co-ordinates of the focus (S) are $x = d$, $y = 0$, the square of the distance of any point P $(x'y')$ from it (Art. 6)

$$= (x' - d)^2 + y'^2;$$

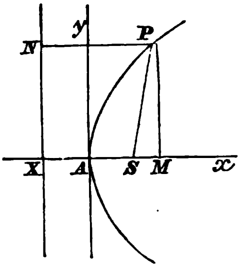
but, if $(x'y')$ be a point on the curve,

$$y'^2 = 4dx';$$

$$\text{hence, } SP^2 = (x' - d)^2 + 4dx' = (d + x')^2,$$

$$\text{or } SP = d + x'.$$

211. *The distance from the directrix of any point in the parabola is equal to its distance from the focus.*



The equation to the directrix is, (Art. 209),

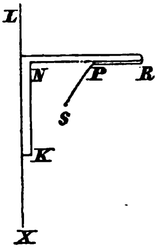
$$x + d = 0, \ldots\ldots\ldots (1)$$

and if a perpendicular be dropped, from any point $P\,(x'y')$ in the curve, upon (1), we have (Art. 43)

$$PN = x' + d = SP.$$

This property is analogous to that proved for central curves (Art. 155), since we shewed (Art. 196) that the parabola might be regarded as an ellipse, whose eccentricity $(e) = 1$.

212. The property proved in the last Article will enable us to describe a parabola mechanically, by means of a ruler and cord. For let a ruler RNK, right-angled at N, slide along a line LX, and let a cord whose length is $= NR$ be fastened at R, and a point S; and while NK slides along LX, let a pencil P be moved so as to keep a portion of the string stretched against RN; then P will trace out a parabola, for the distance PN will be always equal to SP. LX will be the directrix and S the focus.



213. The double ordinate through the focus may be found by putting $x = d$ in the equation $y^2 = 4dx$; then

$$y^2 = 4d^2, \text{ or } y = \pm 2d;$$

hence, the double ordinate $= 4d$. This quantity, as in the case of the ellipse, is called the *Latus Rectum* of the curve.

The latus rectum of the ellipse or hyperbola

$$= \frac{2b^2}{a} \text{ (Art. 156)} = \frac{2(2ad \mp d^2)}{a} \text{ (by Art. 196)},$$

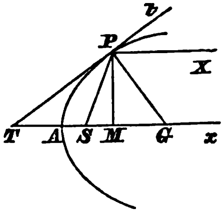
$$= 4d,$$

when a becomes infinite, or the curve passes into a parabola.

214. *The tangent and the normal at any point make equal angles with the focal distance of the point and the line drawn through it parallel to the axis.*

Draw PX parallel to Ax; then the equation to SP, since it passes through $(x'y')$, $(d, 0)$ is

$$y = \frac{y'}{x'-d}(x-d), \quad . \quad . \quad . \quad (1)$$



and to PX, $y - y' = 0$; . . . (2)

hence, the equation to PG, the bisector of the angle between (1) and (2), is

$$\frac{-y(x'-d)+y'(x-d)}{\{y'^2+(x'-d)^2\}^{\frac{1}{2}}} = -(y-y'); \quad (3)$$

making $y = 0$ in (3), we have, since (Art. 210)

$$y'^2 + (x'-d)^2 = (x'+d)^2,$$

$$AG = x = 2d + x';$$

hence, (Art. 205) PG is the normal at P. Hence also it may be seen, that $SP = ST = SG$.

It will be seen that this is a modification of the property proved for the ellipse (Art. 158); for we may suppose the line PX to be in the direction of another focus H, at an infinite distance, and the angles tPX and SPT correspond to the angles which the tangent makes with the two focal distances in the ellipse.

215. *To find the polar equation to the parabola, the focus being the pole.*

Let $SP = \rho$, angle $PSM = \theta$; then (Art. 210)

$$\rho = d + x' = d + AM = 2d + SM = 2d + \rho\cos\theta,$$

or $\rho = \dfrac{2d}{1 - \cos\theta}$,

is the polar equation required.

The polar equation to the ellipse may easily be reduced to this form by the method of Art. 196.

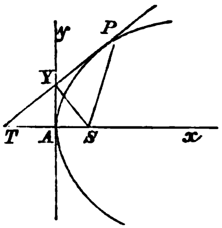
216. *To find the locus of the extremities of perpendiculars dropped from the focus on the tangent.*

The equation to the tangent in terms of its inclination to the axis, is (Art. 201)

$$y = mx + \frac{d}{m};$$

hence, the equation to a line SY, drawn through the focus $(d, 0)$, and perpendicular to this tangent, is

$$y = -\frac{1}{m}(x - d).$$



If we combine these equations in any way, the variables in the result will represent the co-ordinates of Y. Our object is to eliminate m, which quantity particularizes the tangent; hence, subtracting, we have

$$\left(m + \frac{1}{m}\right)x = 0,$$

and, therefore, $x = 0$ is the equation to the required locus, which evidently represents the axis of y, or the tangent at the vertex. This equation may be obtained from the corresponding equation for central conics, as in Article 196.

217. *To find the length of the perpendicular from the focus on the tangent.*

If $SY = p$ (fig. Art. 216), since the perpendicular is drawn from $S(d, 0)$ on the line

$$yy' - 2d(x + x') = 0,$$

and we do not pass over the line to reach the origin, we have

$$p = -\frac{-2d^2 - 2dx'}{\sqrt{y'^2 + 4d^2}}, \text{ (Art. 43).}$$

But $y'^2 + 4d^2 = 4dx' + 4d^2 = 4d(d + x')$.

Hence, $p = \sqrt{d(d + x')}$,

or $p^2 = d\rho$, if $\rho = SP$.

If angle $YSx = \alpha$, we have

$$SY = ST \sin STY,$$

$$\text{or } p = -\rho \cos \alpha \text{ (Art. 214);}$$

hence, substituting for ρ, we have $p = -\dfrac{d}{\cos \alpha}$, and the equation to the tangent, when the focus is origin, may be written in the form of Art. 20,

$$x \cos \alpha + y \sin \alpha + \frac{d}{\cos \alpha} = 0.$$

218. *Any focal chord is perpendicular to the line joining its pole with the focus.*

By definition, the directrix is the polar of the focus, and, therefore, by Art. 208, the polar of any point in the directrix will pass through the focus. Hence, we may assume the pole of any focal chord to be a point $(-d, y')$ in the directrix, and the equation to the chord is (Art. 207)

$$yy' = 2d(x - d), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

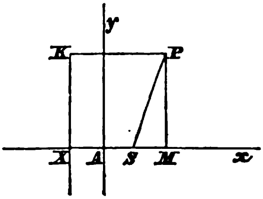
and the equation to a perpendicular to this line, through the focus $(d, 0)$, is

$$y = -\frac{y'}{2d}(x - d), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and when $x = -d$ in (2), $y = y'$, or the line represented by (2) passes through the point $(-d, y')$, which is the pole of the chord; hence the truth of the proposition.

219. *To shew that the locus of a point* (P), *whose distance from a given point* (S) *is equal to its distance* (PK) *from a given straight line* (KX), *is a parabola.*

This is the converse of what was proved Art. 211, where it was shewn that the distance of any point in the parabola from the focus is equal to its distance from the directrix.



Draw SX perpendicular to KX, and make $SA = AX$ $(= d)$, then A is a point in the locus; take the indefinite straight lines Ax and Ay as rectangular axes, and let AM $(= x)$ and MP $(= y)$ be the co-ordinates of the point P; then

$$SP^2 = PK^2,$$

$$\text{or } SM^2 + MP^2 = (AM + AX)^2;$$

$$\text{hence } (x - d)^2 + y^2 = (x + d)^2,$$

$$\text{or } y^2 = 4dx,$$

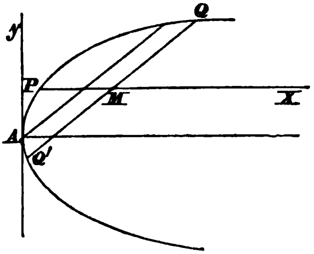
which is the equation to a parabola of which KX is the directrix, A the vertex, Ax the axis, and S the focus.

DIAMETERS.

220. We saw (Art. 116) that there is only one system of rectangular axes which will give the equation to the parabola under the form $y^2 = Lx$, also that, in this case, the axis of x is a diameter, for it bisects all chords parallel to the axis of y, and the axis of y is a tangent (Art. 193) at the extremity of that diameter. We shall now consider the other diameters of the parabola and shew what the equation becomes when it is referred to any one of them and the tangent at its vertex.

Important

221. *To find the locus of the middle points of any system of parallel chords.*



We shall proceed as in the case of central curves. Let QQ' be one of the chords, M $(x'y')$ its middle point, and let the equation to QQ' be

$$\frac{y-y'}{s} = \frac{x-x'}{c} = l.$$

Then for the distance (l) from $(x'y')$ of the points of section of the chord and parabola $(y^2 = 4dx)$, we have

$$(sl+y')^2 - 4d(cl+x') = 0,$$

the roots of which equation (MQ, MQ') are equal and of opposite signs; hence (Appendix) the coefficient of $l = 0$,

$$\text{or } sy' - 2dc = 0,$$

and since s and c are the same for all the chords, this relation holds for the ordinates of all the middle points, or the locus required is

$$sy - 2dc = 0,$$

which represents a straight line (PX) parallel to Ax.

If m be the tangent of the angle that the chords make with the axis, $m = \frac{s}{c}$, and the equation becomes

$$y = \frac{2d}{m}.$$

Cor. It is evident, as in central curves, that the tangent at P, the extremity of PX, is parallel to the chords; for the ordinate y' of $P = \frac{2d}{m}$, and the equation to the tangent is

$$yy' = 2d\,(x + x'),$$

or $$y = m\,(x + x'),$$

which is the equation to a line parallel to the chords.

222. As in central curves, the line which bisects any system of parallel chords is called a *diameter* of the parabola, and the chords are called the *ordinates* of the diameter. We have seen above, that, if the equation to one of the chords be

$$y = mx + c,$$

the equation to the corresponding diameter is

$$y = \frac{2d}{m}.$$

Hence, *all diameters of a parabola are parallel to the axis*, and conversely, all straight lines parallel to the axis may be considered as diameters, for by giving a suitable value to m in the equation $y = \frac{2d}{m}$, y may receive any value we please.

223. *The polar of any point in a diameter is parallel to the ordinates of that diameter.*

We saw (Art. 209), that this is true when the diameter is the axis of the parabola; let $y = \frac{2d}{m}$ be the equation to any diameter, then (Art. 207), the equation to the polar of any point $(x'y')$ in it is

$$yy' = 2d\,(x + x'),$$

and since $y' = \frac{2d}{m}$, this equation becomes

$$y = m\,(x + x'),$$

which is the equation to a line parallel to the chords which the diameter in question bisects. If the point $(x'y')$ be the extremity A' of the diameter (fig. Art. 225), the equation

$$yy' = 2d\,(x + x')$$

represents the tangent at A'; hence, *the tangent at the extremity of any diameter is parallel to the ordinates of that diameter*, as we saw in Art. 221.

224. Since all diameters of a parabola are parallel, it cannot have *conjugate* diameters: it has, however, properties which correspond to conjugate diameters in central curves, and which may be foreseen by regarding the parabola as deduced from the ellipse by the method of Art. 196.

If we refer the ellipse to any diameter and the tangent at its vertex, as axes, the equation will be

$$y^2 = \frac{b'^2}{a'^2}(2a'x - x^2), \quad . \quad . \quad . \quad (1)$$

which results from writing $x - a'$ for x in the equation to the ellipse referred to any two conjugate diameters $2a'$ and $2b'$. Now this equation is of the same form as the equation to the ellipse when the vertex is origin, the major axis the axis of x, and a tangent at the vertex the axis of y. Hence, regarding the parabola as an elongated ellipse, we may conceive (fig. Art. 225), that any diameter $A'X$ has a conjugate at an infinite distance, parallel to the tangent $A'Y$, and we foresee that the equation to the parabola, when referred to $A'X$, $A'Y$ as axes, will be in the same form $(y^2 = Lx)$ as when it referred to the axis of the curve and the tangent at its vertex as axes.

For suppose fig. Art. 225 to represent part of an ellipse, A' the extremity of a diameter and the origin in equation (1), and suppose that $A'X$ meets Ax in the centre C, and there is a diameter CD $(= b')$ conjugate to CA' $(= a')$; let $SA' = d'$; then (Arts. 177, 152), we have

$$SA'.HA' = CD^2, \text{ or } d'(2a - d') = b'^2;$$

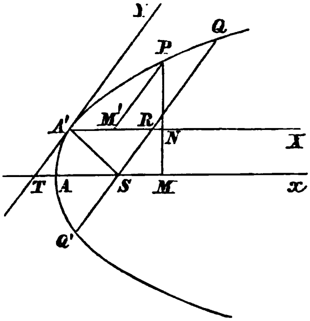
hence equation (1) may be written

$$y^2 = \left(2d' - \frac{d'^2}{a}\right) \cdot \left(2x - \frac{x^2}{a'}\right) \frac{a}{a'}.$$

Now, when the centre is removed to an infinite distance, both a and a' become infinite; also $a : a' :: \sin CA'A : \sin CAA'$, which is a ratio of equality when $A'C$ becomes parallel to Ax, as it does when their point of intersection (C) becomes infinitely distant; hence in this case equation (1) becomes

$$y^2 = 4d'x.$$

We shall, in the next Article, prove this property independently.



225. If we transfer the origin to the extremity $A'(x'y')$ of one of these diameters, we have, writing $x + x'$ for x and $y + y'$ for y in the equation $y^2 = 4dx$,

$$(y + y')^2 = 4d(x + x'),$$

$$\text{or } y^2 + 2y'y = 4dx, \quad . \quad . \quad (1)$$

since $y'^2 = 4dx'$. If we now preserve the axis of x, and take a new axis of y, ($A'Y$) inclined at an angle θ to the axis of x, then

the old $y = PN = PM' \sin\theta$,

the old $x = A'N = A'M' + PM' \cos\theta$,

$A'M'$ and PM' being the new co-ordinates of the point P, when $A'X$, $A'Y$ are axes. Hence, writing $y \sin\theta$ for y and $x + y\cos\theta$ for x in (1), the equation becomes

$$y^2 \sin^2\theta + 2y'y \sin\theta = 4d(x + y\cos\theta),$$

$$\text{or } y^2 \sin^2\theta + y(2y' \sin\theta - 4d\cos\theta) = 4dx. \quad . \quad . \quad . \quad (2)$$

Now, in order that (2) may reduce to the form $y^2 = Lx$, we must have

$$2y' \sin\vartheta - 4d\cos\theta = 0,$$

$$\text{or } \tan\theta = \frac{2d}{y'};$$

but this is the tangent of the angle which the tangent at $A'(x'y')$ makes with the axis of x; hence when we use a diameter and the tangent at its vertex as co-ordinate axes, the equation to the parabola will be

$$y^2 = \frac{4d}{\sin^2\theta}x,$$

where θ is the inclination of the axes.

$$\text{But } \frac{d}{\sin^2\theta} = d\operatorname{cosec}^2\theta = d(1+\cot^2\theta)$$

$$= d\left(1+\frac{y'^2}{4d^2}\right) = d\left(1+\frac{x'}{d}\right)$$

$$= d + x' = SA',$$

and the equation to the parabola referred to any diameter, and the tangent at its vertex is

$$y^2 = 4d'x,$$

where d' is the distance of the origin from the focus. This includes the case of the axis of the parabola and a tangent at its vertex.

226. The quantity $4SA'$ is called the *parameter* of the diameter which passes through A', and, when the diameter is the axis of the parabola, it is sometimes called the *principal parameter*, as well as the *latus rectum*. In all cases it equals the double ordinate through the focus; for draw QQ' through the focus parallel to the tangent at A'; then $A'R = ST = SA'$, (Art. 214),

$$\text{and } QR^2 = 4SA'.A'R = 4SA'^2,$$

$$\text{or } QR = 2SA', \text{ and } QQ' = 4SA'.$$

227. Since the equation to a parabola referred to *any* diameter and the tangent at its vertex is of the same form as when the diameter is the axis, it follows that every property

which has been proved for the latter system is true for all the others, so long as the property does not depend upon the inclination of the axes. For example, if we have, for the equation to the parabola referred to such axes, $y^2 = 4d'x$, we shall find, exactly as in Art. 198, that the equation to the tangent at any point $(x'y')$ is

$$yy' = 2d'(x + x');$$

hence, as before, the subtangent is double the abscissa; also the equation to a line joining the points of contact of two tangents drawn from any point $(x'y')$ is

$$yy' = 2d'(x + x'),$$

and so on.

228. *Tangents at the extremities of any chord will intersect in the diameter of which the chord is an ordinate.*

Let the parameter of the diameter be $4d'$; then, taking the diameter and the tangent at its extremity as axes, the equation to the tangent will be (fig. Art. 225),

$$\pm yy' = 2d'(x + x')$$

according as we take $Q\,(x'y')$, or $Q'(x', -y')$, as the point of contact. In each case, when $y = 0$, x has the same value $= -x'$, or the tangent meets the diameter produced in the same point.

This follows from Art. 221; for, if $y = mx + b$ be the equation to any chord, and $(x'y')$ the point of intersection of the tangents at its extremities, we have also for the equation to the chord (Art. 206),

$$yy' = 2d\,(x + x'),$$

the equation to a line which makes with the axis of x an angle whose tangent $= \dfrac{2d}{y'}$.

Hence $\frac{2d}{y'} = m$, or $y' = \frac{2d}{m}$, and therefore (Art. 221), $(x'y')$ is a point on the diameter of which the chord $(y = mx + b)$ is an ordinate.

EXAMPLES IX.

1. If PSp be any parameter of a parabola whose focus is S and latus rectum L, prove that $4SP \, . \, Sp = L(SP + Sp)$.

2. Find the equation to the normal to a parabola, in terms of its inclination to the axis of the curve.

3. The rectangle contained between two ordinates y_1, y_2 of a parabola, $y^2 = 4dx$, is equal to d^2; to find the magnitudes of y_1 and y_2, the distance between them being $= d$.

4. The tangents to a parabola, drawn from the same point in the directrix, are at right angles.

5. There is only one point in the parabola, at which the focal distance is perpendicular to the tangent.

6. The tangent at any point of a parabola will meet the directrix and latus rectum produced in two points equidistant from the focus.

7. Two equal parabolas have a common axis; a straight line touching the interior and bounded by the exterior, will be bisected in the point of contact.

8. Find a parabola which shall touch a circle at a given point, its axis being coincident with a given diameter.

9. To prove that the area of a triangle inscribed in a parabola is equal to

$$\frac{1}{8d}(y' \sim y'') \, . \, (y'' \sim y''') \, . \, (y''' \sim y'),$$

where y', y'', y''' are the ordinates of the vertices of the triangle, $y^2 = 4dx$ being the equation to the curve.

10. If a parabola intersect a circle in four points, to prove that the ordinates of the points of intersection which lie on one side of the axis of the parabola are together equal to the sum of the ordinates of the points of intersection which lie on the other side of the axis.

11. Two tangents are drawn to a parabola at points whose co-ordinates are a, b, a', b'. To find the point in which they intersect.

12. In the parabola, of which the equation is $y^2 = 4dx$, two tangents are drawn at points of which the abscissæ are in the ratio of $1 : \mu$. To find the equation to the locus of their intersection.

13. From points in the exterior of two equal parabolas having the same axis, tangents are drawn to the interior one; they will touch it at the extremities of diameters whose distance from one another is constant.

14. Three parabolas having their axes parallel, intersect; shew that the three chords passing through their points of contact pass through one point.

15. Two tangents to a parabola make angles whose tangents are a', a'' with the axis; find the equation to the tangent at the extremity of the diameter of which the chord of contact is an ordinate.

16. Find the locus of the middle points of chords passing through any fixed point, and adapt the proof to (i) the focus, (ii) the vertex, and (iii) the foot of the directrix of a parabola.

17. If A be the vertex, S the focus, and PSp the focal chord of a parabola, prove that the rectilinear triangle PAp varies as the square root of the distance Pp.

18. If a line be drawn from the foot of the directrix of a parabola, making an angle 45^0 with the axis, it will touch all parabolas having the same axis and directrix.

19. Draw a normal at the extremity of the latus rectum of the parabola whose equation is $y^2 = 4d(x - d)$, and find its distance from the origin of co-ordinates.

20. Lp is a normal to the parabola at L, the extremity of the latus rectum, meeting the parabola again in p. Shew that the diameter in

which the tangents at L and p intersect, passes through the other extremity of the latus rectum.

21. Two ordinates to a parabola ($y^2 = 4dx$) meet the axis in points equidistant from the focus. If the vertex be joined with the point where one of the ordinates meets the parabola; find the equation to the locus of the point where this line intersects the other ordinate.

22. Two tangents are drawn to a parabola, making angles θ, θ' with the axis. Prove that (i) if $\sin\theta \,.\, \sin\theta'$ be constant, the locus of the intersection of the tangents is a circle, whose centre is in the focus; (ii) if $\tan\theta \,.\, \tan\theta'$ be constant, the locus is a straight line perpendicular to the axis; (iii) if $\cot\theta + \cot\theta'$ be constant, the locus is a straight line parallel to the axis; (iv) if $\cot\theta - \cot\theta'$ be constant, the locus is a parabola equal to the original parabola.

23. A series of triangles are constructed on a given base, their vertices being in a line parallel to the base; shew that the perpendiculars through the extremities of the base to the sides of these triangles will intersect in a parabola whose latus rectum is the distance between the lines.

24. If from any point P of a parabola three lines be drawn, cutting the axis in F, G, H; PG being a normal at P, and the angle FPG being equal to HPG; prove $SG^2 = SF \,.\, SH$.

25. To find the area of a triangle included between the tangents to parabolas $y^2 = 4dx$, $y^2 = 4\delta x$, at points, the common abscissa of which is a, and the portion of the ordinate intercepted between the two curves.

26. To find the magnitude of the ordinate of such a point in a parabola ($y^2 = 4dx$) that the intercepts on the axes of co-ordinates of a tangent drawn to the curve at this point may be equal to each other.

27. The three altitudes of any triangle described about a parabola all pass through a single point in the directrix.

28. To find the distance of the vertex and focus from the tangent in terms of the inclination of the tangent to the axis of x.

29. Find the locus of the centre of the circle which shall always touch a given circle and a given right line.

30. From the vertex of a parabola a straight line is drawn inclined at an angle 45^0 with the tangent at any point; find the equation to the curve which is the locus of their intersection.

31. In the focal distance *SP* take *Sp* equal to the ordinate *PM*. Find the equation to the curve traced out by the point *p*.

32. From two points in the diameter of a parabola two pairs of tangents are drawn to the curve; the trigonometrical tangents of the inclination of one pair to the axis are a_1, a_2, and of the other a_3, a_4; to prove that

$$\frac{1}{a_1}+\frac{1}{a_2}=\frac{1}{a_3}+\frac{1}{a_4}.$$

33. The vertex of a parabola is taken for the centre of a given circle; to find the equation to a line touching both circle and parabola.

34. If from any point *Q* of the line *BQ*, which is perpendicular to the axis *CAB* of a parabola whose vertex is *A*, *QP* be drawn parallel to the axis to meet the parabola in *P*; shew that, if *CA* be taken $= AB$, the locus of the intersection of *AQ* and *CP* is the original curve.

35. If a perpendicular to a focal chord be drawn through the focus, it will pass through the intersection of the tangents at the extremities of the chord.

36. If a triangle be formed by three tangents to a parabola, and chords be drawn joining the points of contact, the triangle so formed by the chords is double of that formed by the tangents.

37. In the sides *AB*, *AC* of a given triangle *ABC* take two points *M*, *N*, and in the line joining them take a point *P*, such that

$$BM : AM :: AN : NC :: PM : PN;$$

find the locus of *P*.

38. Given the radius vector at any point of a parabola and the angle it makes with the curve; find the latus rectum and the position of the vertex.

39. Shew that the parameter belonging to any diameter of a parabola varies inversely as the square of the sine of the angle at which the corresponding ordinates are inclined to it.

40. If from the focus of a parabola lines be drawn to meet the tangents at a given angle, to prove that the locus of their points of intersection will be that tangent to the parabola, the inclination of which to the axis is equal to the given angle.

41. The abscissa and double ordinate of a parabola are h and k, and the diameters of the circumscribed and inscribed circles are D and d. Prove that $D + d = h + k$.

42. If PQ be a chord of a parabola which is a normal at P, and the tangents at P and Q intersect in a point T, shew that PT is bisected in the directrix.

43. To find the equation to all parabolas which are touched by the straight lines $y = \pm \frac{x}{2}$.

44. To find the equation to the normal of a parabola, which is inclined at any given angle to the axis of the curve.

45. Two normals to a parabola ($y^2 = 4dx$) are always at right angles to each other; to find the locus of their intersection.

46. To find the equations to all the common chords of the two curves $y^2 = 2cx - x^2$, $y^2 = 4dx$.

47. To prove that a series of circles of which the centres are in a parabola, and which pass through the focus, all touch the directrix.

48. If from the focus of a parabola as centre and with the focal distance of a point in the parabola as radius, a circle be described, to prove that the intersections of the tangent and the normal to the parabola at the point, with the axis, will lie in the circumference of the circle.

49. If PT, QT are two tangents to a parabola at P, Q, and S the focus, $SP \cdot SQ = ST^2$.

50. From a point P, the concourse of two tangents to a parabola PQ, PQ', $PABC$ is drawn meeting the curve in A, C, and QQ' in B. PA, PB, PC are in harmonical progression.

51. To find the locus of intersection of perpendiculars from the focus on the normal.

52. If two tangents be drawn to a parabola, to prove that a third tangent, parallel to the chord joining the points of contact, will bisect the parts of the other tangents which are included between their point of intersection and their points of contact.

53. The normal at one extremity of the latus rectum is a polar; to find the co-ordinates of its pole.

54. The abscissæ of two points in a parabola, reckoned along the axis, are x, $3x$, and the corresponding focal distances r, $2r$; to find the position of the former of these points.

55. Any number of parabolas are described having the same vertex and axis, and any straight line is drawn at right angles to the common axis. If any points whatever in this line be taken as poles, to prove that all the polars belonging to all the parabolas will intersect in a single point.

56. To find the locus of the centre of a circle which shall pass through a given point and touch a given line.

57. Find the condition that any line $\frac{x}{a} + \frac{y}{b} = 1$, should touch the parabola $y^2 = 4dx$.

58. The centre of an ellipse coincides with the vertex of a parabola, and the axis major of the ellipse is perpendicular to the axis of the parabola; required the proportion of the axes of the ellipse that it may cut the parabola at right angles.

59. Transform the equation to the tangent to the ellipse in the form $y = mx + \sqrt{a^2m^2 + b^2}$ into the corresponding equation for the parabola.

60. Given a point where a parabola intersects a given diameter, and also the parameter of that diameter; shew that the locus of the vertex is an ellipse.

SECTION XI.

Methods of Abridged Notation. Various forms of the Equation of the Second Order.*

229. We shall in this chapter give some further illustration of the use of the abridged system of notation to which the student has been already introduced in Arts. 45–48, and which, from its extreme conciseness, will often enable us to solve problems that are almost impracticable by the ordinary methods.

230. We have shewn that, if

$$\alpha = 0, \quad \beta = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be the equations to two lines, the equations

$$\alpha - k\beta = 0, \quad \alpha + k\beta = 0 \quad . \quad . \quad . \quad (2)$$

will represent two straight lines, so drawn, that the perpendiculars dropped from any point on them upon the lines (1), are to one another in the ratio of k to 1.

The lines $\alpha = 0$, $\beta = 0$, are usually called 'the line (α),' 'the line (β),' and their point of intersection 'the point (α, β)'; so the point of intersection of the lines (2) would be 'the point $(\alpha - k\beta,\ \alpha + k\beta)$'. Many problems occur, which may be solved by means of this notation, where it is not necessary to introduce any limitation as to the forms of the equations and the position of the origin. In cases, however, where we wish to make use of the reasoning of Arts. 43—48, it will be necessary to fix the form of the equation, and consider attentively the relative positions of the origin and the lines with which we have to do, before we commence.

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* This subject is discussed at greater length in Salmon's *Conic Sections*, of which considerable use has been here made.

231. The lines (2) represent two systems of straight lines, and any particular line in either of these systems may be selected by giving k the particular value which it has for that line; to do this we require some other condition besides the one that it passes through the intersection of the lines (1), for this is common to them all.

The simplest condition would be that it should pass through some given point.

Suppose then that α, β, become α', β', when the co-ordinates of the given point are substituted in them for x and y; then since this point lies in $\alpha - k\beta = 0$, we have

$$\alpha' - k\beta' = 0,$$

$$\text{and} \therefore \alpha - \frac{\alpha'}{\beta'} . \beta = 0, \text{ or } \alpha\beta' - \alpha'\beta = 0,$$

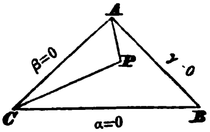
the required equation, in which k, the arbitrary constant, does not appear.

232. We have shewn (Art. 47 Cor.) that, when $k = 1$, equations (2) become

$$\alpha - \beta = 0, \quad \alpha + \beta = 0 \quad . \quad . \quad . \quad . \quad (3)$$

and represent the bisectors of the angles between the lines (α) and (β). A little consideration will shew that the lines ($\alpha - k\beta$), ($k\alpha - \beta$) are equally inclined to the line ($\alpha - \beta$); as are also ($\alpha + k\beta$) and ($k\alpha + \beta$) to the line ($\alpha + \beta$).

Ex. *If* ABC *be a triangle, and* AP, CP, *be so drawn that the sines of the angles* CAP, PAB, PCB, *and* PCA, *are the consecutive terms of a proportion, shew that the locus of* P *is a conic section.*



Taking the origin of co-ordinates *within* the triangle, so as to avoid any difficulty about the sign of the arbitrary constant k, the equations to CB, AC, AB

may be written $\alpha = 0$, $\beta = 0$, $\gamma = 0$, and those of *AP* and *PC* will therefore by the hypothesis be

$$\beta - k\gamma = 0,\ \beta - \frac{1}{k}\alpha = 0,$$

and therefore, eliminating k, we have for the locus of *P*

$$\beta^2 - \alpha\gamma = 0. \quad . \quad . \quad . \quad (4)$$

Now since α, β, γ are all of one dimension in x and y, this equation is of two dimensions, and is therefore *some* conic section; the particular conic section may be determined by writing α, β, γ at full length in (4), and then finding the coefficients of x^2, y^2, and xy in the usual manner, (Art. 106).

233. *If* (α), (β), (γ), *be three straight lines, and we can find three constants*, λ, μ, ν, *such that*

$$\lambda\alpha + \mu\beta + \nu\gamma = 0, \quad . \quad . \quad . \quad (5)$$

identically, *then all the three straight lines* (α), (β), (γ), *pass through one point.*

For the co-ordinates of the point (α, β) make both α and β vanish, and therefore, if these values are substituted for x and y in (5), they will give us

$$\nu\gamma = 0, \text{ or } \gamma = 0,$$

and, consequently, this point lies also in the third line (γ); that is to say, all three lines pass through the same point.

Ex. (1) *The three lines that bisect the angles of a triangle meet in a point.*

Taking the origin of co-ordinates within the triangle, let $\alpha = 0$, $\beta = 0$, $\gamma = 0$ be the equations to the three sides; then the equations to the straight lines bisecting the angles are

$$\alpha - \beta = 0,\ \beta - \gamma = 0,\ \gamma - \alpha = 0,$$

and these by (5) evidently intersect in one point.

In this case, $\lambda = \mu = \nu = 1$.

Ex. (2) If, through the angular points of a triangle, there be drawn any three lines meeting in a point, then three lines, drawn through the same angles, equally inclined to the bisectors of the angles, will also meet in a point.

The first three lines may be represented by

$$l\alpha - m\beta = 0,\ m\beta - n\gamma = 0,\ n\gamma - l\alpha = 0,$$

putting $\frac{m}{l}$ for k &c.

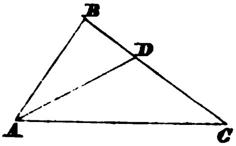
Then (Art. 282), the equations to the other three will be

$$m\alpha - l\beta = 0,\ n\beta - m\gamma = 0,\ l\gamma - n\alpha = 0,$$

and multiplying these by $\frac{1}{ml}$, $\frac{1}{nm}$, and $\frac{1}{ln}$, respectively, they become

$$\frac{\alpha}{l} - \frac{\beta}{m} = 0,\quad \frac{\beta}{m} - \frac{\gamma}{n} = 0,\quad \frac{\gamma}{n} - \frac{\alpha}{l} = 0,$$

and therefore we see that these also pass through one point.



Ex. (3) The lines joining the angular points of a triangle with the middle points of the opposite sides intersect in one point. Let *ABC* be the triangle and the origin as before within it: let $\alpha = 0$, $\beta = 0$, $\gamma = 0$ be the equations to *BC*, *AC*, and *AB*. Then if *D* be the middle point of *BC*, the equation to *AD* is

$$\beta - \frac{\sin CAD}{\sin BAD}\gamma = 0;$$

but $\frac{\sin CAD}{\sin C} = \frac{CD}{AD}$, and $\frac{\sin BAD}{\sin B} = \frac{BD \text{ or } CD}{AD}$,

and therefore

$$\frac{\sin CAD}{\sin BAD} = \frac{\sin C}{\sin B},$$

and the equation to AD becomes

$$\beta \sin B - \gamma \sin C = 0.$$

Similarly the equations to the other two lines are

$$\gamma \sin C - \alpha \sin A = 0,$$
$$\alpha \sin A - \beta \sin B = 0,$$

and these three lines evidently pass through one point.

Ex. (4) It may be shewn in the same manner, that, if AD be perpendicular to BC, its equation is

$$\beta \cos B - \gamma \cos C = 0,$$

and that those of the other perpendiculars are

$$\gamma \cos C - \alpha \cos A = 0,$$
$$\alpha \cos A - \beta \cos B = 0,$$

and these three pass through the same point.

Ex. (5) If three circles intersect, their chords of contact all pass through one point.

Let $(x - a)^2 + (y - b)^2 = r^2$ be one of the circles; then expanding this equation, we have,

$$x^2 + y^2 - 2(ax + by) + a^2 + b^2 - r^2 = 0,$$

which may be written

$$x^2 + y^2 + \alpha = 0; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

α being linear in x and y.

Similarly, the equations to the other circles may be written,

$$x^2 + y^2 + \beta = 0, \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$x^2 + y^2 + \gamma = 0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now, subtracting (2) from (1), we have the straight line

$$\alpha - \beta = 0,$$

which is satisfied for the points of intersection of (1) and (2), and is therefore their chord of contact. Similarly it may be shewn that the equations to the two other chords of contact are

$$\beta - \gamma = 0,$$
$$\gamma - \alpha = 0;$$

and these three evidently pass through one point.

234. By reasoning precisely similar to that made use of in the case of straight lines, we see that, if

$$S = 0, \qquad S' = 0, \quad . \quad . \quad . \quad (1)$$

be two conic sections, (S and S' being of two dimensions in x and y) then

$$S - kS' = 0; \quad . \quad . \quad . \quad . \quad (2)$$

(where k is some arbitrary constant,) since it is also of two dimensions, is also a conic section; and since the co-ordinates of the points of intersection of equations (1) evidently satisfy equation (2), therefore (2) represents a system of conic sections, passing through the points of intersection of (1).

Any particular conic of this system may be obtained by giving k its corresponding value for that conic; and this value of k must be determined from the particular condition that distinguishes the conic from all others of the same system; as for instance, that it "passes through a given point," or "touches a given line," &c.

Ex. Find the equation to the conic section that passes through the intersection of the circles

$$x^2 + y^2 - 4x - 8y = 28, \quad . \quad . \quad . \quad (1)$$
$$x^2 + y^2 = 4, \quad . \quad . \quad . \quad (2)$$

and through the centre of (1).

Equation (1) may be written

$$(x - 2)^2 + (y - 4)^2 = 48,$$

and the co-ordinates of its centre are therefore (2, 4).

The equation to the required conic will be of the form

$$(x-2)^2+(y-4)^2-48-k(x^2+y^2-4)=0,$$

and since it passes through the point (2, 4), we have,

$$-48-k\,.\,16=0,$$

$$\therefore k=-3,$$

and the conic becomes

$$(x-2)^2+(y-4)^2-48+3(x^2+y^2-4)=0,$$

$$\text{or } 4x^2+4y^2-4x-8y=40,$$

$$\text{or } x^2+y^2-x-2y=10;$$

a circle, the co-ordinates of whose centre are $(\frac{1}{2}, 1)$, and whose radius is $\frac{3}{2}\sqrt{5}$.

235. Suppose that $S'=\alpha\beta$,

where α and β are of one dimension in x and y, and consequently (S'), though apparently a conic section, really represents two straight lines, then

$$S-k\alpha\beta=0, \quad . \quad . \quad . \quad . \quad (1)$$

will represent a conic, two of whose chords of intersection with the conic S are

$$\alpha=0, \qquad \beta=0.$$

This can be instantly deduced from the previous Article; for since the conic (S') has degenerated into two straight lines, the conic (1) which passes through the intersections of (S) and these straight lines must have the lines for two of its six chords of intersection with (S).

It can be also shewn independently, thus; since the co-ordinates of the points of intersection of (α) and (S) make these expressions vanish simultaneously, they also make the expression $S-k\alpha\beta$ vanish, and therefore satisfy equation (1). Hence the conic (1), passing through the intersections of (α) and (S), has (α) for its chord of intersection with (S).

Similarly (β) may be shewn to be a chord of intersection of (1) and (S).

236. If $\alpha = \beta$ then S' becomes of the form α^2, and the equation $S' = 0$, or $\alpha^2 = 0$, represents two straight lines which coincide. Then, by following out the same train of reasoning as that used above, we see that, since the lines (α) and (β) coincide, the conic

$$S - k\alpha^2 = 0,$$

has one chord of contact instead of six chords of intersection with (S), and that this chord of contact is the line ($\alpha = 0$). This could also have been proved independently, as above. ($S - k\alpha^2$) is said to have a *double contact* with (S). The annexed figures will shew the position of the conics of Arts. 235, 236.

Fig. 1.

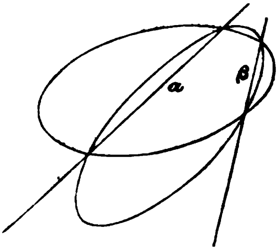
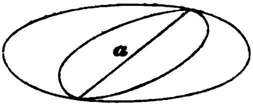


Fig. 2.



237. If S also, as well as S', can be split up into two linear factors, then each of the two conics degenerates into two straight lines, and our equation $S - kS' = 0$ becomes of the form

$$\alpha\gamma - k\beta\delta = 0; \quad . \quad . \quad . \quad (1)$$

where α, β, γ, δ are all of one dimension in x and y; and we see that (1) represents a conic section passing through the four points of intersection of the degenerate conics, or, in other words, circumscribing the quadrilateral formed by these four straight lines.

As this is a proposition of very great use, we shall however prove it independently; it may be stated thus:

If $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$, *be the equations to the four sides of a quadrilateral* taken in order, *then the equation*

$$\alpha\gamma - k\beta\delta = 0, \quad . \quad . \quad . \quad (2)$$

represents a system of conics circumscribing the quadrilateral.

For the co-ordinates of the point (α, β) make α and β simultaneously vanish, and therefore make the expression $\alpha\gamma - k\beta\delta$ vanish, and therefore satisfy equation (2). This corner of the quadrilateral lies therefore in (2). Similarly it may be shewn that the points (β, γ), (γ, δ), (δ, α), or the other three corners lie in (2), and therefore that (2) (which being of two dimensions in x and y must be *some* conic) is one of a system of conics circumscribing the quadrilateral.

Ex. *Find the equation to a conic section passing through five given points.*

Let (a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), (a_5, b_5) be the five given points.

Then

$$\begin{aligned}
(a_1 - a_2)(y - b_1) - (b_1 - b_2)(x - a_1) &= 0,\\
(a_2 - a_3)(y - b_2) - (b_2 - b_3)(x - a_2) &= 0,\\
(a_3 - a_4)(y - b_3) - (b_3 - b_4)(x - a_3) &= 0,\\
(a_4 - a_1)(y - b_4) - (b_4 - b_1)(x - a_4) &= 0,
\end{aligned}$$

represent four straight lines intersecting each other in the points (a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), and therefore forming a quadrilateral which has these points for its corners.

Let the above four lines be represented by

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0;$$

then the conics which circumscribe the quadrilateral are all contained in the equation

$$\alpha\gamma - k\beta\delta = 0. \quad . \quad . \quad . \quad . \quad (1)$$

Now let α, β, γ, δ, become α', β', γ', δ', when a_5, b_5 are substituted for x and y in them.

Then since (a_5, b_5) is a point in (1), by the hypothesis, we have

$$\alpha'\gamma' - k\beta'\delta' = 0, \text{ and } \therefore k = \frac{\alpha'\gamma'}{\beta'\delta'};$$

and (1) becomes

$$\alpha\gamma\beta'\delta' - \alpha'\gamma'\beta\delta = 0,$$

the equation to the required conic.

Written at full, k would be

$$\frac{\{(a_1-a_2)(b_5-b_1)-(b_1-b_2)(a_5-a_1)\}\,\{(a_3-a_4)(b_5-b_3)-(b_3-b_4)(a_5-a_3)\}}{\{(a_2-a_3)(b_5-b_2)-(b_2-b_3)(a_5-a_2)\}\,\{(a_4-a_1)(b_5-b_4)-(b_4-b_1)(a_5-a_4)\}}.$$

238. It is worthy of remark that, if $\alpha\gamma - k\beta\delta = 0$ be a conic, then the intersections of the lines

$$\left.\begin{aligned}\alpha \pm \lambda\delta = 0\\ \gamma \pm \frac{k}{\lambda}\beta = 0\end{aligned}\right\}, \text{ or } \left.\begin{aligned}\alpha \pm \lambda\beta = 0\\ \gamma \pm \frac{k}{\lambda}\delta = 0\end{aligned}\right\},$$

where the signs are taken upper together or lower together, and λ is arbitrary, are points on the curve. We have already had an example of this in the case of supplemental chords (Arts. 79, 184).

239. If in the equation

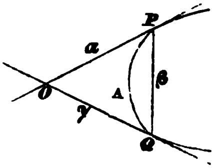
$$\alpha\gamma - k\beta\delta = 0,$$

$\beta = \delta$, then it becomes

$$\alpha\gamma - k\beta^2 = 0;$$

and we see that, owing to the two opposite sides of the quadrilateral approaching to and ultimately coinciding with each other, instead of two chords of intersection, β and δ, of the circumscribing conic, we have only one chord, which will be one of contact; for the lines α, γ will then each pass through two consecutive points in the circumscribing conic, and will therefore be tangents to it.

The general position of the three lines is given in the figure.



The ordinary equation to central conics will furnish an example of this; for it may be written

$$(x - a)(x + a) + \frac{a^2}{b^2} y^2 = 0,$$

when $x - a = 0$, $x + a = 0$ are two tangents, and $y = 0$ is their chord of contact.

Ex. *To find the equation to a parabola referred to two tangents as axes.*

Let PAQ be the parabola; OP, OQ two tangents, $OP = b$, $OQ = a$, and let O be taken as origin, and OP, OQ as axes of y and x.

Then the equations to OP, OQ, PQ are

$$x = 0,\ y = 0,\ \frac{x}{a} + \frac{y}{b} - 1 = 0,$$

and therefore the equation to the conic PAQ is

$$xy - k\left\{\frac{x}{a} + \frac{y}{b} - 1\right\}^2 = 0, \quad . \quad . \quad . \quad (1)$$

and that this may be a parabola we have (Art. 106),

$$\frac{4k^2}{a^2b^2} = \left(1 - \frac{2k}{ab}\right)^2,$$

$$\text{and therefore } k = \frac{ab}{4}.$$

The equation consequently becomes

$$xy - \frac{ab}{4}\left\{\frac{x}{a} + \frac{y}{b} - 1\right\}^2 = 0,$$

which may be written

$$\frac{x}{a}+\frac{y}{b}-1=\pm 2\sqrt{\frac{xy}{ab}},$$

and therefore $\sqrt{\frac{x}{a}}\pm\sqrt{\frac{y}{b}}=\pm 1$, . . . (2)

the general form of the equation to the parabola referred to two tangents as axes.

240. We observed (Art. 17), that the general equation of the first degree has in reality only two independent constants, though apparently containing three, and that, consequently, a straight line could only be subjected to two independent conditions, since these would give two relations between the constants, which would suffice to determine them. Similarly, the general equation of the second degree (Art. 100) possesses but five independent constants, and a conic section can, consequently, only be subjected to five independent conditions. If fewer conditions than five are given, the resulting equation must contain the same number of arbitrary constants as there are conditions wanting. For instance, in the Example to the last Article, the conic is limited, first, to touch the two lines OP, OQ, from whence we have equation (1), where there are three constants, k, a, b, undetermined. If we introduce a third condition, viz. that the conic shall be a parabola, we get rid of k, and obtain equation (2), where a and b are still arbitrary. If the lengths of a and b be given, we have five conditions, and the conic is completely determined.

241. We may remark here that, if α, β, γ be linear functions of x and y, any equation of the first degree may be written in the form

$$\lambda\alpha+\mu\beta+\nu\gamma=0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where λ, μ, ν are constant quantities; for (1) is evidently of the first degree, and it contains the same number of independent constants (viz. two) as the general equation

$$Ax+By+C=0, \text{ (Art. 17)},$$

and is therefore equally capable of representing any particular straight line.

In the same manner the equation to any conic may be written in the form

$$A\alpha^2 + B\alpha\beta + C\beta^2 + D\alpha\gamma + E\beta\gamma + F\gamma^2 = 0, \quad . \quad . \quad (2)$$

where A, B, C, D, E, F are constants; for (2) is evidently of the second degree, and contains the same number of disposable constants (viz. five) as the general equation of the second degree, and is therefore equally capable of representing any particular conic.

If (α), (β), (γ) be written in the form $x \cos\alpha + y \sin\alpha = p$, we have, in the above equations, relations between the distances from the *three* lines (α), (β), (γ) of any point in the line which the equation represents, exactly as, in rectangular co-ordinates, we have relations between the distances from *two* straight lines, viz. the axes.

242. The theorems of Art. 241 do not hold when (α), (β), (γ) meet in a point; for since in that case there is a certain point whose co-ordinates make $\alpha = 0$, $\beta = 0$, $\gamma = 0$, simultaneously, those co-ordinates will also satisfy (1) and (2), and therefore (1) and (2) will be always limited to pass through the point in which (α), (β), (γ) intersect, and do not represent *any* straight line or *any* conic. Also, if the lines (α), (β), (γ) are parallel, it may be easily seen, by writing the equations at length, that (1) can only represent straight lines parallel to them. And, in this case also, (2) will not represent *any* conic; for the lines (α), (β), (γ) take the forms

$$Ax + By + C = 0, \quad Ax + By + C' = 0, \quad Ax + By + C'' = 0,$$

and therefore equation (2) takes the form

$$P(Ax + By)^2 + Qx + Ry + S = 0,$$

which represents (Art. 101) parabolas and parallel straight lines only.

243. *To find the equation to the conic circumscribing the triangle whose sides are* (α), (β), (γ).

Let the equation be written in the form of equation (2) Art. 241; then, since the conic passes through the point (α, β), the equation must be satisfied by making $\alpha = 0$, $\beta = 0$; but this gives

$$F\gamma^2 = 0;$$

hence we must have $F = 0$; for since (γ) does not pass through the point (α, β), γ is not $= 0$ at that point. Similarly, it may be shewn that $A = 0$, and $C = 0$, and the equation becomes

$$B\alpha\beta + D\alpha\gamma + E\beta\gamma = 0.$$

This equation may be written in the form

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0,$$

and evidently involves two independent arbitrary constants only, the conic having already fulfilled three conditions.

244. *To find the equation to a conic inscribed in the triangle whose sides are* (α), (β), (γ).

Let (α) be a tangent to the curve represented by the general equation (2) Art. 241; then, making $\alpha = 0$, we have

$$C\beta^2 + E\beta\gamma + F\gamma^2 = 0, \quad . \quad . \quad . \quad (1)$$

which represents a conic passing through the intersection of (α) and the original conic, and will, in general, when combined with (α), give two values of x and y, which refer to the two points of intersection of (α) with the original conic. But if (1) be a perfect square, *i.e.* represents two coincident straight lines, these two points will become coincident, or (α) will touch the original conic. In this case (1) will be of the form

$$m^2\beta^2 - 2mn\beta\gamma + n^2\gamma^2 = 0,\text{*}$$

---

* The line (α) would also meet the curve in one point if it were a tangent to (1), in which case (1) might still remain a conic; but we evidently can introduce no condition with this notation that this should be the case *generally*, since (1) is independent of (α).

and, by similar reasoning for the other two sides (β) and (γ), we see that the general equation must take the form

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2ln\alpha\gamma - 2lm\alpha\beta = 0. \quad . \quad (2)$$

We might evidently give whatever signs we please to the three last terms of equation (2). We have written it with the negative signs, because the conic does not then degenerate into two straight lines, as it evidently will in some cases. For example, if they be written with positive signs, the equation becomes

$$(l\alpha + m\beta + n\gamma)^2 = 0,$$

which represents two coincident straight lines, evidently fulfilling equally well the conditions of the problem, since they meet the lines (α), (β), (γ), each in one point only. The equation

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0,$$

may be easily seen to be equivalent to equation (2); it contains two disposable constants only, as it ought, for the conic has been made to fulfil three out of five conditions (Art. 240).

245. *To find the condition that a straight line should touch the conic.*

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0. \quad . \quad . \quad . \quad (1)$$

Let the straight line be (Art. 241) written in the form

$$\lambda\alpha + \mu\beta + \nu\gamma = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We have from (1)

$$l\alpha - m\beta - n\gamma - 2\sqrt{mn\beta\gamma} = 0, \quad . \quad . \quad . \quad . \quad (3)$$

and dividing (2) by λ, (3) by l, and subtracting, we have

$$\left(\frac{\mu}{\lambda} + \frac{m}{l}\right)\beta + \left(\frac{\nu}{\lambda} + \frac{n}{l}\right)\gamma + \frac{2\sqrt{mn}}{l}\sqrt{\beta\gamma} = 0. \quad . \quad . \quad (4)$$

Now (4) will, in general, when combined with (2), give two values of x and y, which will refer to the two points of intersection of (1) and (2), but when (4) is a complete square, it will represent one straight line, and will therefore, when combined with (2), give one point of intersection only; in which case (2) is a tangent to (1). Now if (4) be a complete square, we have

$$\left(\frac{\mu}{\lambda}+\frac{m}{l}\right)\left(\frac{\nu}{\lambda}+\frac{n}{l}\right)=\frac{mn}{l^2},$$

from whence

$$\frac{l}{\lambda}+\frac{m}{\mu}+\frac{n}{\nu}=0, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the required condition.

Ex. If (α), (β), (γ) be the tangents to an ellipse at A, A', B, its ordinary equation may be written in the form

$$\sqrt{l(x-a)}+\sqrt{m(x+a)}+\sqrt{n(y-b)}=0, \quad . \quad . \quad . \quad (6)$$

and it may be easily seen, by writing the equation as equation (2) Art. 244, and introducing the condition that the coefficients of xy, x, and y should vanish, that

$$l=-b,\ m=b,\ n=-2a.$$

Now the equation $y+b=0$ may be written

$$b(x-a)-b(x+a)-a(y-b)=0, \quad . \quad . \quad . \quad (7)$$

which evidently fulfils condition (5), as it ought, since it represents the tangent at B'.

246. *To find the locus of the centre of a conic section.*

When a conic has been subjected to four conditions, its equation will involve one arbitrary constant only; if then we take the two equations of Art. 108 for determining its centre, we may, in general, eliminate this arbitrary constant between them, and the result will be the locus required.

Ex. A conic is described, touching two straight lines at given distances (a and b) from their point of intersection; find the locus of its centre.

Taking the two lines as axes, the equation to the conic is, by Art. 239,

$$xy - k\left(\frac{x}{a} + \frac{y}{b} - 1\right)^2 = 0;$$

hence, the equations for the centre are (Art. 108)

$$y - \frac{2k}{a}\left(\frac{x}{a} + \frac{y}{b} - 1\right) = 0;$$

$$x - \frac{2k}{b}\left(\frac{x}{a} + \frac{y}{b} - 1\right) = 0.$$

Eliminating k, we have for the required locus,

$$ay - bx = 0,$$

a line bisecting the chord of contact.

EXAMPLES X.

1. IF from a point P, without a quadrilateral, perpendiculars be let fall upon the four sides, such that the ratio of the product of those on opposite sides is constant, then the locus of P is a conic circumscribing the quadrilateral.

Let $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$, be the equations to the four sides, written in the form, $x \cos \alpha + y \sin \alpha - p = 0$.

Then the lengths of the perpendiculars from the point $P(x\ y)$ upon them are respectively

$$\alpha,\ \beta,\ \gamma,\ \delta,$$

and therefore, by the hypothesis,

$$\frac{\alpha\gamma}{\beta\delta} = k,$$

where k is constant; and since this is true for every position of P, therefore the equation

$$\alpha\gamma - k\beta\delta = 0,$$

is the locus of P, and is, we know, a conic section circumscribing the quadrilateral.

2. If two conics have each a double contact with a third conic, the chords of intersection of the first two, as well as the chords of contact of the first two with the third, all pass through one point.

$$\text{Let}\quad S - k\alpha^2 = 0, \quad . \quad . \quad (1) \qquad S - \lambda\beta^2 = 0, \quad . \quad . \quad (2)$$

be two conics, which have a double contact with the conic $S = 0$.

Then for the intersection of (1) and (2), we have

$$k\alpha^2 - \lambda\beta^2 = 0,$$

$$\text{and} \therefore\ \alpha - \beta\sqrt{\frac{\lambda}{k}} = 0, \quad . \quad . \quad (3) \qquad \alpha + \beta\sqrt{\frac{\lambda}{k}} = 0; \quad . \quad . \quad (4)$$

which being linear in x and y are therefore the chords of intersection of (1) and (2), and they manifestly pass through the same point as the lines α, β.

If $\lambda = k$, then the straight lines (3) and (4) bisect the angles between α and β.

3. If (α), (β), (γ) be the sides of a triangle ABC opposite to A, B, C, and points D, E, F be taken in (α), (β), (γ), to shew that the sides of the triangle DEF may be represented by

$$\alpha + n\beta + \frac{\gamma}{m} = 0, \qquad \frac{\alpha}{n} + \beta + l\gamma = 0, \qquad m\alpha + \frac{\beta}{l} + \gamma = 0,$$

where l, m, n are constants.

Without limiting the position of the origin, or the *forms* of α, β, γ, we may write the equations

to AD, $\beta + l\gamma = 0$,
to BE, $\gamma + m\alpha = 0$,
to CF, $\alpha + n\beta = 0$;

then we have for the points

$$D, \left.\begin{array}{r}\alpha = 0\\ \beta + l\gamma = 0\end{array}\right\}, \quad E, \left.\begin{array}{r}\beta = 0\\ \gamma + m\alpha = 0\end{array}\right\}, \quad F, \left.\begin{array}{r}\gamma = 0\\ \alpha + n\beta = 0\end{array}\right\};$$

hence, the equation to DE is, since it passes through D, of the form

$$\beta + l\gamma + k\alpha = 0, \quad . \quad . \quad . \quad (1)$$

and, since it passes through E, we have

$$-lm\alpha + k\alpha = 0, \text{ or } k = lm,$$

and the equation to DE becomes

$$\beta + l\gamma + lm\alpha,$$

which may be written

$$m\alpha + \frac{\beta}{l} + \gamma = 0.$$

Similarly it may be shewn that the equations to the other sides may be written as stated above.

4. If the lines (α) and (β) be parallel, shew that the line $(\alpha + k\beta)$ is parallel to each of them, k being any constant.

5. From the angles A, B, C of any triangle are drawn three straight lines AA', BB', CC', bisecting the angles; through A, B, C are drawn three straight lines perpendicular to AA', BB', CC', to meet BC, CA, AB, produced, in G, H, K; G, H, K are in one straight line.

6. ABC is a triangle, D and E are points within the triangle, such that the angle $ABE = CBD$, and $BCD = ACE$; shew that $BAD = CAE$.

7. In Ex. 3, find the condition that AD, BE, CF may pass through one point.

8. If $\alpha = 0$, $\beta = 0$, $\gamma = 0$ be three sides of a triangle, the equations being of the form $x \cos \alpha + y \sin \alpha - p = 0$, and the origin being within the triangle, interpret the equations

$$\beta + \gamma - \alpha = 0, \qquad \alpha + \gamma - \beta = 0, \qquad \alpha + \beta - \gamma = 0.$$

9. The four angles of a quadrilateral $ABCD$ are bisected by four straight lines; the bisectors of A, B meet in E, of B, C in F, of C, D in G, of A, D in H. Prove that the directions of FH and EG pass through the intersection of the directions of AD, BC, and AB, CD respectively.

10. To find the equation to the conic section which passes through the five points $1, -1$; $-2, 3$; $3, 2$; $-1, -3$; $2, 1$.

11. Shew that the equation of Art. 243 becomes

$$\beta\gamma \sin A + \alpha\gamma \sin B + \alpha\beta \sin C,$$

when (α), (β), (γ) are of the form $x \cos \alpha + y \sin \alpha = p$, A, B, C are the angles between $\beta\gamma$, $\alpha\gamma$, $\alpha\beta$, and the conic is a circle.

12. Shew that the equations to the three tangents to the curve, at the angular points of the triangle in Art. 243 are

$$l\gamma + n\alpha = 0, \quad l\beta + m\alpha = 0, \quad m\gamma + n\beta = 0.$$

13. In the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, suppose B to assume different values, all the other coefficients remaining the same; then (i) the conics which it represents are, in general, all described about the same quadrilateral, and (ii) the locus of their centres is another conic whose equation is $2Cy^2 + Ey = 2Ax^2 + Dx$.

14. From the equation to the ellipse, written as in Art. 245, Ex., deduce the equation to the tangent in the form $y = mx \pm \sqrt{a^2m^2 + b^2}$, using condition (5) in the same Art.

15. Find the equation to a parabola which cuts the axes of x and y at distances a, a', b, b' from the origin.

16. Write the equation to the ellipse considered as a conic circumscribing the quadrilateral which is formed by joining the extremities of a pair of conjugate diameters.

17. The locus of the centre of a conic, which cuts the axes as in Ex. 15, is a conic, the co-ordinates of whose centre are $\frac{1}{4}(a + a')$, $\frac{1}{4}(b + b')$.

18. From the equation to a conic circumscribing a quadrilateral, shew that, if the conic be a circle, the opposite angles of the quadrilateral are supplementary.

19. If a conic be inscribed in a triangle, the three straight lines joining the angular points with the points of contact pass through one point.

20. If two conics have two points of contact, to prove that they will not meet in any other point.

21. An ellipse cuts an hyperbola in four points, and touches the asymptotes in A and B; shew that two of the chords of intersection are parallel to AB.

22. If $\alpha=0$, $\beta=0$, $\gamma=0$ be three equations written in the form $x\cos\alpha+y\sin\alpha-p=0$, interpret the equations

$$\sqrt{\alpha\sin A}+\sqrt{\beta\sin B}+\sqrt{\gamma\sin C}=0,$$
$$\sqrt{\alpha\cos A}+\sqrt{\beta\cos B}+\sqrt{\gamma\cos C}=0,$$
$$\sqrt{\alpha}+\sqrt{\beta}+\sqrt{\gamma}=0,$$

the origin being within the triangle.

23. If a conic touch the sides of the triangle of Ex. 22, in D, E, F, then (i) any one of the lines AF, BE, CD, drawn from the opposite angles, passes through the intersection of tangents drawn to the conic at the points where the other two cut it; and (ii) if p, q, r be perpendiculars drawn from the point where these three lines meet (Ex. 19), the equation to the conic is

$$\sqrt{\frac{\alpha}{p}}+\sqrt{\frac{\beta}{q}}+\sqrt{\frac{\gamma}{r}}=0.$$

24. Find the diameter of a circle described about a semi-ellipse bounded by its axis minor.

25. From a point O, two straight lines are drawn, touching a conic in H and K. A straight line $MQRT$ is drawn, parallel to one of these tangents, cutting the other in M, the curve in Q, T, and the chord HK in R; to shew that $MQ\,.\,MT=MR^2$.

26. If a conic be inscribed in a triangle, and touch two sides at the point of their bisection, shew that the line joining the centre and the third point of contact will pass through the opposite angular point.

27. If a conic be inscribed in a quadrilateral, the line joining its point of contact with two opposite sides passes through the intersection of the diagonals.

28. From the angles of a triangle ABC lines are drawn through a given point O within the triangle to meet the opposite sides in E, F, G; FG, GE, EF are produced to meet BC, CA, AB, in P, Q, R; shew that P, Q, R lie in one straight line.

29. If the asymptotes of an hyperbola ($xy = \frac{1}{4}m^2$) coincide with the conjugate diameters of an ellipse, which has a double contact with the hyperbola, then $4a^2b^2 = m^4$, where a, b are the semi-conjugate diameters.

30. Shew, geometrically, that, when the conic is a circle, the equation of Art. 244 becomes

$$\alpha^{\frac{1}{2}} \cos \tfrac{1}{2}A + \beta^{\frac{1}{2}} \cos \tfrac{1}{2}B + \gamma^{\frac{1}{2}} \cos \tfrac{1}{2}C.$$

31. Shew that the tangents to a conic, drawn at the angular points of an inscribed triangle, will meet the opposite sides in points which all lie in one straight line.

32. A conic is inscribed in a triangle whose sides are (α), (β), (γ); shew that the equations to the lines joining the points of contact are

$$l\alpha + m\beta - n\gamma = 0, \quad l\alpha + n\gamma - m\beta = 0, \quad m\beta + n\gamma - l\alpha = 0.$$

33. Shew that the three conics

$$\alpha\gamma - \frac{ln}{m^2}\beta^2 = 0, \quad \alpha\beta - \frac{lm}{n^2}\gamma^2 = 0, \quad \beta\gamma - \frac{mn}{l^2}\alpha^2 = 0,$$

meet in a single point, and that, if the constants l, m, n are connected by the equation $\frac{\lambda}{l} + \frac{\mu}{m} + \frac{\nu}{n} = 0$, where λ, μ, ν are fixed quantities, the locus of that point is a conic circumscribing the triangle (α), (β), (γ). Explain this by a geometrical figure.

SECTION XII.

Miscellaneous Propositions.

247. We shall now shew how some of the principal properties that have been proved in the foregoing sections may be deduced at once from the general equation of the second order.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \quad . \quad . \quad . \quad . \quad (1)$$

Let the equation to a line meeting the curve be

$$\frac{y-y'}{s} = \frac{x-x'}{c} = l; \quad . \quad . \quad . \quad . \quad (2)$$

where $s = \sin\alpha$ or $\dfrac{\sin\alpha}{\sin\omega}$, $c = \cos\alpha$ or $\dfrac{\sin(\omega-\alpha)}{\sin\omega}$, according as (Art. 27) the axes are rectangular or oblique. Then, from (1) and (2), we have, to determine the distances from $(x'y')$ of the points of intersection, an equation

$$Pl^2 + Ql + R = 0, \quad . \quad . \quad . \quad . \quad (3)$$

$$\text{where } P = Ac^2 + Bcs + Cs^2,$$

$$Q = (2Ax' + By' + D)c + (2Cy' + Bx' + E)s,$$

$$R = Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F.$$

Now, as in Arts. 167, &c., if $(x'y')$ be the middle point of the chord, Q must $= 0$, and we obtain, for the equation to the diameter of which (2) is an ordinate,

$$(2Ac + Bs)x + (2Cs + Bc)y + Dc + Es = 0. \quad . \quad . \quad (4)$$

If $Q = 0$ for every value of s and c, *i.e.* if $(x'y')$ be the centre of the curve, we must have

$$2Ax' + By' + D = 0, \quad 2Cy' + Bx' + E = 0,$$

which are the equations of Art. 108.

248. If $(x'y')$ be on the curve, we have $R = 0$ in (3), and we may, as before, Arts. 135, &c. obtain, for the equation to the tangent at $(x'y')$,

$$\frac{y-y'}{x-x'} = -\frac{2Ax'+By'+D}{2Cy'+Bx'+E}. \quad . \quad . \quad . \quad . \quad (5)$$

Multiplying up, and remembering that $(x'y')$ is a point on (1), equation (5) will become

$$(2Ax'+By'+D)x+(Bx'+2Cy'+E)y+Dx'+Ey'+2F=0, \quad . \quad . \quad (6),$$

and by reasoning such as we have used in the case of the several curves, it will be seen that this is also the equation to the polar of any point $(x'y')$. From these equations all the ordinary theories of polars, tangents, and conjugate diameters may be deduced.

249. The rectangle on the segments of the chord (2) will be equal to the product of the roots of equation (3), and therefore

$$= \frac{R}{P} = \frac{Ax'^2+Bx'y'+Cy'^2+Dx'+Ey'+F}{Ac^2+Bcs+Cs^2},$$

and if another chord be drawn through the same point $(x'y')$ and its direction be determined by s' and c', quantities corresponding to s and c above, the rectangle on its segments

$$= \frac{Ax'^2+Bx'y'+Cy'^2+Dx'+Ey'+F}{Ac'^2+Bc's'+Cs'^2};$$

hence the ratio of the rectangles is

$$Ac^2+Bcs+Cs^2 \; : \; Ac'^2+Bc's'+Cs'^2,$$

a ratio which does not depend upon the point $(x'y')$ and which remains the same as long as the chords make, respectively, the same angles with the axes; hence, *if* QQ′, RR′ *be two chords of a conic, and* P *their point of intersection, the ratio* PQ . PQ′ : PR . PR′ *is not altered by moving each chord parallel to itself, and so shifting the position of* P *in any manner.*

We have already had a particular instance of this theorem in the case of central conics referred to any pair of conjugate diameters; for their equation may be written

$$\frac{y^2}{(a' - x)(a' + x)} = \frac{b'^2}{a'^2},$$

where the two positions of the point $(x'y')$ are the foot of the ordinate and the centre.

250. *If the equation to a conic be*

$$Ax^2 + Bxy + Cy^2 + F = 0, \quad . \quad . \quad . \quad . \quad (1)$$

the asymptotes are represented by

$$Ax^2 + Bxy + Cy^2 = 0 \quad . \quad . \quad . \quad . \quad (2)$$

The equation to the tangent to (1) is (Art. 248), putting D and E each $= 0$,

$$2Ax'x + By'x + Bx'y + 2Cy'y + 2F = 0; \quad . \quad . \quad . \quad (3)$$

Dividing by y', and then making $y' = \infty$, (3) becomes

$$2Ax\frac{x'}{y} + Bx + By\frac{x'}{y'} + 2Cy = 0. \quad . \quad . \quad . \quad (4)$$

But (4) evidently passes through the origin, and therefore

$$\frac{x'}{y'} = \frac{x}{y},$$

and equation (4) becomes

$$Ax^2 + Bxy + Cy^2 = 0, \quad . \quad . \quad . \quad . \quad (5)$$

and is the equation to the asymptotes.

By Art. 57, the tangent of the angle between the lines represented by (5) is

$$= \frac{(B^2 - 4AC)^{\frac{1}{2}}}{A + C},$$

an expression which, by Arts. 108, 112, does not change when the equation is transformed either by shifting the origin or moving the axes through any angle.

251. *To find the polar equation to the tangent of a conic section, the focus being the pole, and the angular coordinate* (α) *of the point of contact being given.*

It will be seen, by Arts. 157, 215, that the focal polar equation to any conic may be written

$$\frac{C}{\rho} = 1 - e\cos\theta, \quad . \quad . \quad . \quad . \quad (1)$$

where C = half the latus rectum, and $e = 1$ for the parabola; the left hand focus being the pole in the ellipse, and the right hand in the hyperbola; then (Art. 51, Ex. 1) the polar equation to a straight line may be written

$$\frac{C}{\rho} = A\cos\theta + B\sin\theta. \quad . \quad . \quad . \quad . \quad (2)$$

At the intersection of (1) and (2) we have

$$1 - (A + e)\cos\theta = B\sin\theta,$$

and $\therefore 1 - B^2 - 2(A+e)\cos\theta + \{(A+e)^2 + B^2\}\cos^2\theta = 0.$ (3)

Now, if (2) be a tangent, (3) has equal roots,

$$\text{and } \therefore (A+e)^2 = \{(A+e)^2 + B^2\}.(1 - B^2),$$

$$\therefore 0 = B^2 - B^2\{(A+e)^2 + B^2\},$$

$$\text{or } 1 - B^2 = (A+e)^2;$$

hence, substituting for B^2 in (3), we have

$$(A+e)^2 - 2(A+e)\cos\theta + \cos^2\theta = 0;$$

$$\text{hence } A + e = \cos\alpha, \text{ and } \therefore 1 - B^2 = \cos^2\alpha;$$

$$\therefore A = \cos\alpha - e, \; B = \sin\alpha,$$

and therefore, substituting in (2), the equation to the tangent becomes

$$\frac{C}{\rho} = \cos(\theta - \alpha) - e\cos\theta.$$

Ex. 1. *To find the angle that the tangent makes with the axis of* x.

Make ρ infinite; then the radius vector is parallel to the tangent, and we have

$$\cos(\theta - a) = e\cos\theta,$$

$$\text{whence } \tan\theta = \frac{e - \cos a}{\sin a}.$$

In the parabola, $e = 1$, which gives $\theta = \frac{a}{2}$, and therefore $SP = ST$, as it ought (Art. 214).

Ex. 2. *In the parabola, if* SP, SQ *be two radii vectores, and* PT, TQ *tangents at* P *and* Q, *then* SP . SQ = ST², *and* ST *bisects the angle* PSQ.

Ex. 3. *The locus of the intersection of tangents at points which subtend a given angle* (ϕ) *at the focus of a conic is a conic which has the same focus and directrix, and whose eccentricity* $= \dfrac{e}{\cos\frac{1}{2}\phi}$.

252. *To find the equation to a conic, when the tangent and normal at any point are the axes of* y *and* x.

Since the axis of y is a tangent at the origin, when $x = 0$ in the general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

the values of y become each $= 0$,

$$\therefore E = 0,\ F = 0,$$

and the equation is reduced to

$$Ax^2 + Bxy + Cy^2 + Dx = 0,$$

the axis of x being any line drawn through the point of contact.

253. The following is a method very commonly used for finding the equation to the tangent at any point of a curve.

Suppose $(x'y')$, $(x''y'')$ to be two points near to one another in the curve; then the equation to the line passing through these points is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'), \quad . \quad . \quad . \quad . \quad (1)$$

and if we suppose the point $(x''y'')$ to move up to $(x'y')$ and approach indefinitely near to it, (1) will be a tangent at $(x'y')$. The fraction $\frac{y'' - y'}{x'' - x'}$ will, in this case, assume the indeterminate form $\frac{0}{0}$, and we must therefore endeavour to find its real value. If the equation to the curve be $f(xy) = 0$, where $f(xy)$ denotes an expression involving x and y, we shall have, since $(x'y')$, $(x''y'')$ are on the curve,

$$f(x'y') = 0, \; f(x''y'') = 0,$$

$$\text{and } \therefore f(x''y'') - f(x'y') = 0. \quad . \quad . \quad . \quad (2)$$

From which equation we may generally find a value of $\frac{y'' - y'}{x'' - x'}$, which does not become indeterminate when $(x''y'')$ approaches to $(x'y')$.

We will suppose, for example, that the curve is an ellipse or hyperbola; then for (2) we shall have

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} - 1 - \left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1\right) = 0,$$

hence

$$\frac{y''^2 - y'^2}{x''^2 - x'^2} = -\frac{b^2}{a^2},$$

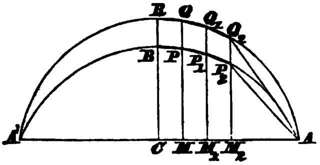
$$\text{and we have } \frac{y'' - y'}{x'' - x'} = -\frac{b^2(x'' + x')}{a^2(y'' + y')},$$

hence, when $(x''y'')$ approaches indefinitely near to $(x'y')$, we have

$$\frac{y''-y'}{x''-x'} = -\frac{b^2x'}{a^2y'},$$

and, substituting in (1), we shall evidently obtain the equation of Art. 135.

254. *To find the area of the ellipse.*



Let the common ordinates of an ellipse and the circle on its transverse axis be QPM, $Q_1P_1M_1$, &c.; then since (Art. 125)

$$PM : QM :: P_1M_1 : Q_1M_1 :: b : a,$$

$$\therefore PM + P_1M_1 : QM + Q_1M_1 :: b : a.$$

Also the quadrilaterals PMM_1P_1, QMM_1Q_1 give

$$PMM_1P_1 = \tfrac{1}{2}(PM + P_1M_1)\,MM_1,$$

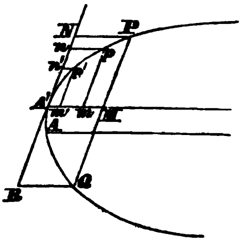
$$QMM_1Q_1 = \tfrac{1}{2}(QM + Q_1M_1)\,MM_1;$$

$$\text{hence, } PMM_1P_1 : QMM_1Q_1 :: b : a,$$

and the same ratio holds for each pair of quadrilaterals; hence, the whole polygon BPP_1P_2A inscribed in the ellipse, is to the whole polygon RQQ_1Q_2A inscribed in the circle, in the ratio of b to a. Now, since this will be true whatever be the number of the sides of the polygons, let us suppose the number to be increased indefinitely; then the areas of the polygons approach indefinitely near to the areas of the ellipse and circle, and we perceive that the area of the ellipse is to the area of the circle as $b : a$; and, since the area of the circle is πa^2, the area of the ellipse $= \pi ab$ or $= \pi a'b' \sin\gamma$ if a', b' be two semi-conjugate diameters and γ be the angle between them (Art. 180).

Cor. For the hyperbola, the same relation obtains between the area contained by the curve and a chord parallel to the conjugate axis, and the corresponding area of an equilateral hyperbola described upon the transverse axis. But this will not enable us to find the area of the hyperbola, unless we first know that of the equilateral hyperbola.

255. *To find the area of the segment of a parabola cut off by any right line.*



Let PQ be the line; draw the diameter $A'M$ bisecting it, $A'N$ the tangent at A; let $p\ (xy)$, $p'\ (x'y')$ be two points near to one another; then if $\angle NA'M = \omega$, the quadrilateral

$$pmm'p' = \tfrac{1}{2}(y + y')(x - x')\sin\omega,$$

$$pnn'p' = \tfrac{1}{2}(x + x')(y - y')\sin\omega,$$

$$\therefore\ \frac{pmm'p'}{pnn'p'} = \frac{(y + y')(x - x')}{(x + x')(y - y')} = \frac{(y + y')^2}{y^2 + y'^2},$$

since $y'^2 = 4a'x'$, $y^2 = 4a'x$ (Art. 225).

Now when p' approaches indefinitely near to p and therefore y' to y, we have

$$\frac{pmm'p'}{pnn'p'} = \frac{(2y)^2}{2y^2} = \frac{2}{1}.$$

Now we may divide the area $A'MP$ into a series of quadrilaterals such as $pmm'p'$, and the area $A'NP$ into a corresponding series $pnn'p'$, and if the number of these figures becomes indefinitely large, the ratio of each of those in the area $A'MP$ to the corresponding one in $A'NP$ will be 2 to 1; hence the sum of one set is to the sum of the other in the same ratio, and we have

$$\text{area } A'MP = 2 \text{ area } A'NP;$$

hence the whole segment $PA'Q$ is equal to $\frac{2}{3}$ of the parallelogram $PNRQ$.

256. *Eccentric angle.*

It is sometimes convenient, in the solution of problems, to express the abscissa and ordinate of any point in the curve in terms of some single variable. In the circle, referred to its centre as origin, if ϕ be the angle which the radius drawn to the point (xy) makes with the axis of x, we have shewn (Art. 95) that we may write $x = r\cos\phi$, $y = r\sin\phi$, thus expressing them both in terms of the single variable ϕ.

In the ellipse, we may assume that

$$x = a\cos\phi,\ y = b\sin\phi,$$

for these assumptions might be made separately, since $x < a$, $y < b$, and when substituted in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

they give

$$\sin^2\phi + \cos^2\phi = 1,$$

and hence they may be made simultaneously.

In the hyperbola we may put

$$x = a\sec\phi,\ y = b\tan\phi,$$

since the secant and tangent may have any values, and these values, when substituted in the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

will give

$$\sec^2\phi - \tan^2\phi = 1.$$

This angle is called the *Eccentric Angle* of the point. We shall leave it to the student to construct the angle geometrically. He will find that, in the ellipse, if the ordinate MP be produced to meet the circle on the major axis in P' the angle $P'CM = \phi$. In the hyperbola, if a tangent MQ be drawn from the foot of the ordinate MP, to the circle on the transverse axis, the angle QCM is the eccentric angle of the point P.

257. *Harmonic Pencils.*

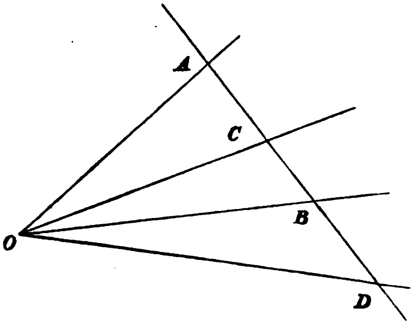
Def. 1. Any four lines meeting in a point form a *pencil*.

Def. 2. A straight line is said to be *harmonically* divided when it is divided into three parts, such that the whole line is to one of the extreme segments, as the other extreme segment is to the middle part.

Def. 3. A straight line drawn across a pencil is called a *transversal*.

Def. 4. A pencil is called *harmonic* if it divides any transversal harmonically.

258. Let OA, OC', OB, OD be any pencil, and let $ACBD$ be a transversal; we shall shew that the ratio $\dfrac{AD}{AC} \div \dfrac{BD}{BC}$ is a constant ratio, in whatever way AD be drawn across the system.



For
$$\frac{AD}{AO} = \frac{\sin AOD}{\sin ADO},$$

$$\frac{AO}{AC} = \frac{\sin ACO}{\sin AOC},$$

$$\therefore \frac{AD}{AC} = \frac{\sin AOD \,.\, \sin ACO}{\sin AOC \,.\, \sin ADO}.$$

Similarly
$$\frac{BD}{BC} = \frac{\sin BOD \,.\, \sin OCB}{\sin ADO \,.\, \sin BOC},$$

$$\therefore \frac{AD}{AC} \div \frac{BD}{BC} = \frac{\sin AOD \,.\, \sin BOC}{\sin AOC \,.\, \sin BOD}, \quad \ldots \quad (1)$$

which is a ratio independent of the position of AD. When the right hand member of equation (1) = 1, the line is *harmonically* divided, and the pencil is *harmonic*. We then have

$$\frac{AD}{AC} = \frac{BD}{BC}; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, if we consider AD, AB, AC as the first, second, and third quantities, respectively, equation (2) asserts that the first is to the third as the difference between the first and second is to the difference between the second and third, and the quantities are therefore in harmonical progression.

259. It may be proved, as above, that the ratios

$$\frac{AD}{AB} \div \frac{CD}{BC}, \quad \frac{AB}{AC} \div \frac{BD}{CD}, \quad . \quad . \quad . \quad (3)$$

are constant.

The constant ratio (1) is called *the anharmonic ratio* of the pencil; but the constant ratios (3) are also sometimes called anharmonic ratios.

260. The lines

$$\alpha = 0, \;\; \beta = 0, \;\; \alpha - k\beta = 0, \;\; \alpha + k\beta = 0,$$

form an harmonic pencil.

Let OA, OB, be the lines, $\alpha = 0$, $\beta = 0$, then, if we suppose the origin somewhere in the angle AOB, the lines $\alpha - k\beta = 0$, $\alpha + k\beta = 0$ will (Art. 47) lie as OC and OD respectively. Then we have

$$\frac{\sin AOC}{\sin BOC} = k = \frac{\sin AOD}{\sin BOD};$$

$$\therefore \frac{\sin AOC \,.\, \sin BOD}{\sin AOD \,.\, \sin BOC} = 1,$$

or, Art. 258,

$$\frac{AD}{AC} = \frac{BD}{BC},$$

and the pencil is harmonic.

261. *Similar Conics.*

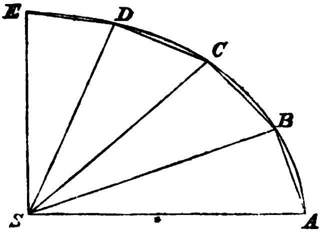
Def. Similar curvilinear figures are figures whose curved boundaries are curvilinear limits of corresponding portions of similar polygons, when the number of sides is increased indefinitely, and their magnitudes, at the same time, diminished indefinitely.

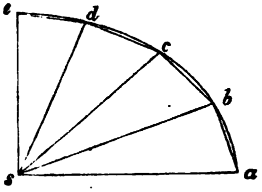
From this definition it follows at once, as a *test* of similar curves, that, one curve is similar to another, when if *any* polygon be inscribed in one, a similar polygon can be inscribed in the other.

262. *Two curves are similar when, any point* S *being taken in the plane of one curve, another point* s *can be found in the plane of the other, such that, any radii* SA, SB *being drawn in the first, radii* sa, sb *can be drawn in the second having the properties that*

$$\angle ASB = \angle asb,$$

$$\text{and } SA : SB :: sa : sb.$$





For let S, s be such points in the planes of the curves $ABCDE$, $abcde$, so that the angles ASB, BSC... are equal to the angles asb, bsc,... and

$$SA : SB : SC ... :: sa : sb : sc ....$$

Then the triangles ASB, asb, &c. are (Euc. VI. 6) similar, and therefore the angles ABC, BCD... are equal to the angles abc, bcd,... also

$$AB : ab :: SB : sb :: BC : bc,$$

$$\therefore\ AB : BC : CD ... :: ab : bc : cd, ...$$

or the parts of the polygons are similar which are bounded by corresponding radii. But this is true however large be the number of the sides of the polygons. Hence, if the number of sides be increased indefinitely, the curves, which are the limits of the polygons, are similar.

263. *If, in the planes of two curves, one such pair of points, as* S *and* s, *can be found, an infinity of other pairs can be found.*

For the existence of one pair proves, as above, the similarity of the curves. Hence, let *any* polygon $ABCDE$ be inscribed in one, and a similar polygon $abcde$ inscribed in the other, and let AS, BS... be drawn to *any* point S. Construct the triangle sab equiangular to the triangle SAB, and join sc, sd....

$$\text{Then } SB : sb :: AB : ab$$
$$:: BC : bc,$$

$$\text{and since } \angle ABC = \angle abc, \text{ and } \angle ABS = \angle abs,$$

$$\therefore\ \angle SBC = \angle sbc,$$

$$\text{and } \therefore\ SBC, sbc \text{ are similar triangles, and}$$

$$SC : sc :: SB : sb$$
$$:: SA : sa,$$

and similarly for $SD : sd$, &c. Hence, since these are *any* polygons, the result is true for *any* radii which may be drawn from S, s to the curves, in the required manner.

264. Curves are said to be *similar and similarly situated* when the proportional radii are parallel.

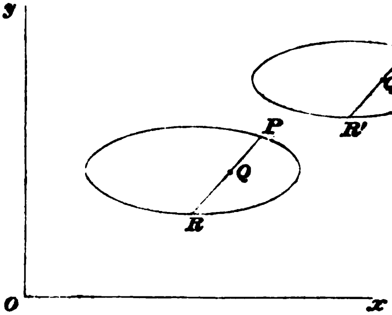
The two fixed points S and s are called *centres of similarity.*

265. *To find the condition that the two conics*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad . \quad . \quad (1)$$

$$A'x^2 + B'xy + C'y^2 + D'x + E'y + F' = 0, \quad . \quad . \quad (2)$$

should be similar and similarly placed.



Let us suppose $Q(x'y')$, $Q'(x''y'')$ to be two points, such as are described above, and suppose PQ, $P'Q'$ to be parallel radii vectores making an angle θ with the axis of x, and let their equations be

$$\frac{y-y'}{s} = \frac{x-x'}{c} = l, \ (3) \quad \frac{y-y''}{s} = \frac{x-x''}{c} = l \ (4).$$

Combining (3) and (4) with (1) and (2) respectively we shall obtain two equations (Art. 247),

$$Pl^2 + Ql + R = 0, \ . \ (5) \quad P'l^2 + Q'l + R' = 0, \ . \ (6)$$

the roots of which are the distances of Q and Q' respectively from the curves. But, by hypothesis, as long as PQ is parallel to $P'Q'$, their ratio is constant, or

$$P'Q' = k \, . \, PQ \text{ suppose.}$$

Hence, if we write kl for l in equation (6), the values of l in (5) and (6) will be the same, and therefore we shall have

$$\frac{P}{P'k^2} = \frac{Q}{Q'k} = \frac{R}{R'}. \quad . \quad . \quad . \quad . \quad (7)$$

Since (Art. 247) R and R' depend only upon the position of Q and Q', and do not change as the lines move about these points, we may assume the ratio of $R : R'$ as constant, and we have (Art. 247), since $s = \sin\theta$, $c = \cos\theta$,

$$\frac{P}{P'} = \frac{A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta}{A'\cos^2\theta + B'\sin\theta\cos\theta + C'\sin^2\theta} = \frac{k^2R}{R'} = \mu \text{ say}, \quad . \quad (8)$$

$$\text{or } A - \mu A' + (B - \mu B')\tan\theta + (C - \mu C')\tan^2\theta = 0, \quad . \quad (9)$$

for *all* values of θ; but this can only be the case when

$$A - \mu A' = 0,\ B - \mu B' = 0,\ C - \mu C' = 0,$$

$$\text{or } \frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}.$$

Hence, if two conics are similar and similarly placed, the coefficients of the highest powers of the variables are the same in both, or differ only by a constant multiplier.

266. We shall now shew that the condition of the previous article is *sufficient* to ensure similarity.

I. The curves must be of the same class; for, if

$$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'} = \mu, \quad . \quad . \quad . \quad . \quad (1)$$

we have

$$B^2 - 4AC = \mu^2(B'^2 - 4A'C'),$$

that is, the quantities $B^2 - 4AC$, $B'^2 - 4A'C'$ must be both positive, both negative, or both zero, or the conics must be both hyperbolas, both ellipses, or both parabolas. If they be central curves, we may, by transferring the origin to the centre of each, obtain their equations in the form

$$Ax^2 + Bxy + Cy^2 + F = 0,$$
$$A'x^2 + B'xy + C'y^2 + F' = 0,$$

A, B, C being the same as before, by Art. 108. From these equations we should have in polar co-ordinates

$$\rho^2 = \frac{-F}{A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta},$$
$$\rho'^2 = \frac{-F'}{A'\cos^2\theta + B'\sin\theta\cos\theta + C'\sin^2\theta},$$

and, if θ have the same value in each of these equations, or the radii vectores be parallel, we shall have the ratio of $\rho : \rho'$ constant, if the condition (1) holds. Hence, the centre of each curve is a point answering the required condition; consequently any number of such points may be found, and the curves are similar.

It will easily be seen, by reference to the polar equation to the parabola (Art. 215), that all parabolas are similar figures, the foci being centres of similarity, and the parallel radii being in the ratio of the latera recta.

267. It will be seen, from the condition of Art. 265, that the axes of two similar and similarly placed conics are parallel; for, by Art. 111, according as we wish to make the axes of co-ordinates parallel to the axes of the one or the other respectively, we shall have to turn them through an angle θ which satisfies the equation

$$\tan 2\theta = \frac{B}{A - C}, \text{ or } \tan 2\theta = \frac{B'}{A' - C'}.$$

But by the condition

$$\frac{B}{A - C} = \frac{B'}{A' - C'},$$

or the angle θ will be the same in each case.

268. If two central conics are similar and similarly placed, we see from the above that all diameters of one are constantly proportional to the parallel diameters of the other. From this it follows again that their eccentricities are the same; for the ratio $\frac{b}{a}$ is the same in each, and therefore $\frac{a^2 - b^2}{a^2}$ or e^2. Also, by reference to the polar equations (Arts. 122, 157), it may be seen, that, conversely, if the axes are parallel and the eccentricities the same, the conics will be similar and similarly placed, having the foci and centres as centres of similarity.

Again, if two hyperbolas have parallel asymptotes, they must be similar and similarly placed; for their axes, which bisect the angles between the asymptotes must be parallel, and their eccentricities, which depend upon $\frac{b}{a}$, the tan of $\frac{1}{2}$ the angle between the asymptotes, must be equal.

269. Two conics will be similar, but not similarly placed, if the proportional radii make a constant angle with each other instead of being parallel; *i. e.* they would fulfil the conditions of Art. 265, and be both similar and similarly placed, if one of them were moved through a given angle (α).

Let equations (1) and (2) Art. 265 represent two conics, similar but not similarly placed, and let conic (1) be turned through such an angle α, that it may be similarly placed with (2). This will, of course, as far as regards the change in its equation, be the same as if the axes were turned through an angle α, and we shall have a new equation,

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \quad . \quad . \quad (3)$$

where the values of $a, b, c, \ldots$ in terms of $A, B, C, \ldots$ and the angle α, may be seen from Art. 111, and, from Art. 112, we have also the following relations which do not involve α,

$$b^2 - 4ac = B^2 - 4AC, \quad a + c = A + C. \quad . \quad . \quad (4)$$

But, since (3) is similar and similarly placed to (2), we have

$$\frac{a}{A'} = \frac{b}{B'} = \frac{c}{C'} = \mu \text{ say};$$

from which equations we have

$$b^2 - 4ac = \mu^2(B'^2 - 4A'C'), \ a + c = \mu(A' + C').$$

Hence,
$$\frac{B'^2 - 4A'C'}{(A' + C')^2} = \frac{b^2 - 4ac}{(a + c)^2},$$

and
$$\therefore \frac{B'^2 - 4A'C'}{(A' + C')^2} = \frac{B^2 - 4AC}{(A + C)^2}$$

is, from (4), the required condition.

270. Since the elimination of one of the variables, between two equations of the second degree, produces generally (Appendix) an equation of the fourth degree, it may be seen that two conics will, generally, meet in four points. When the curves meet in two real points only, it is convenient, for some purposes, to speak of the other two points of intersection as being impossible or infinitely distant, or both. For example, two similar and similarly situated conics S and S' cannot intersect in more than two real points; for S' may be multiplied by such a constant k as to make the equation

$$S - kS' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

linear, and, therefore, combining (1) with either $S = 0$ or $S' = 0$, we shall get but two points of intersection. Hence, it would seem that any other conic

$$S - k'S' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

would be limited to pass through two points only, and therefore (Art. 240) should have three arbitrary constants instead of one; but such an equation as (2) gives us *algebraical* conditions which we cannot always express geometrically unless we consider that (2) passes through the same infinite or impossible points as S and S'. We should then have four conditions, as we ought. And, in fact, (2) does in reality fulfil two other conditions, when S and S' are similar, for it is *similar and similarly placed* with S and S'; for if

$$S = Ax^2 + Bxy + Cy^2 + \&c. \quad . \quad . \quad . \quad . \quad = 0,$$

$$S' = \mu Ax^2 + \mu Bxy + \mu Cy^2 + \&c. \quad . \quad . \quad . \quad = 0,$$

$$S - k'S' = A(1 - k'\mu)x^2 + B(1 - k'\mu)xy + C(1 - k'\mu)y^2 + \&c. = 0,$$

which is a conic similar and similarly placed with S and S'.

If the curves be hyperbolas, their asymptotes are parallel, and, as parallel lines may be supposed to meet at infinity, and each asymptote meets its own curve at infinity, the hyperbolas tend to intersect at two points at an infinite distance.

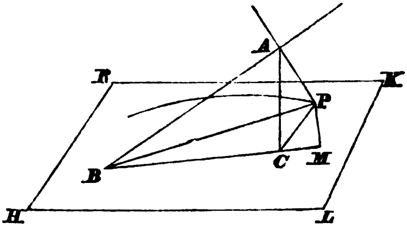
271. *Sections of the Cone.*

The surface described by an indefinite straight line, which is carried round the perimeter of a given circle, always passing through a given point, is called a cone. The circle is called the base of the cone, the fixed point the vertex, and the line joining the vertex and the centre of the base is called the axis. A cone is said to be *right* if the axis is perpendicular to the plane of the base, and *oblique* if the axis is inclined at any other angle to that plane. As the generating line is not limited, the surface of the cone consists of two portions or sheets (fig. Art. 274), perfectly similar, situated on opposite sides of the vertex, and of indefinite extent. It is evident, from the method in which the cone is generated, that every plane parallel to the base will cut the cone in a circle, and that every plane through the axis will cut it in two straight lines, in both which cases, the section will be represented by an equation of the second degree. We shall now shew that the same is the case, in whatever manner the plane cuts the cone, and we shall, for the sake of simplicity, content ourselves with proving this property in the case of right cones only.

The different curves, obtained by cutting a cone by a plane, are called *Conic Sections.*

272. *Every section of a right cone by a plane is a curve of the second degree.*

Obs. The generating line, in a right cone, will always make the same angle with the axis.



Let *HRLK* be a plane; *AB*, a fixed line, the axis of a cone, inclined at an angle a to the plane; *AC* a perpendicular from *A*, the vertex, on the plane; *AP* the generating line, revolving round *AB*, inclined to it at a constant angle β. Then *P*, the extremity of *AP*, will evidently trace out some conic section on the plane, which is supposed to be intercepted between the vertex and the circle round the perimeter of which the generating line is carried.

Draw *PM* perpendicular to *BC* produced; join *BP* and *CP*. Take *C* as origin, *CM* as axis of x, and a perpendicular to it in the plane $HRKL$, as axis of y; let P be the point (xy) and $AB = a$.

$$\text{Then } BP^2 = PM^2 + BM^2$$

$$= y^2 + (x + a\cos\alpha)^2, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

since $\angle ABC = \alpha$; also

$$BP^2 = a^2 + AP^2 - 2a \,.\, AP \,.\, \cos\beta$$

$$= a^2 + (a^2\sin^2\alpha + y^2 + x^2) - 2a\cos\beta\sqrt{a^2\sin^2\alpha + y^2 + x^2}, \;.\; (2)$$

since $AP^2 = AC^2 + PC^2$.

Equating (1) and (2), the equation to the locus of P is

$$a^2\cos^2\alpha + 2ax\cos\alpha = a^2 + a^2\sin^2\alpha - 2a\cos\beta\sqrt{a^2\sin^2\alpha + x^2 + y^2},$$

$$\text{or } \cos\beta\sqrt{a^2\sin^2\alpha + x^2 + y^2} = a\sin^2\alpha - x\cos\alpha,$$

a curve of the second degree.

273. Comparing this equation with the general equation of the second degree

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

we see that in this case

$$A = \cos^2\beta - \cos^2\alpha, \;\; B = 0, \;\; C = \cos^2\beta.$$

Now (Art. 101), the curve is an Hyperbola, Parabola, or Ellipse, according as $B^2 - 4AC > = < 0$, or as

$$-\cos^2\beta\,(\cos^2\beta - \cos^2\alpha) > = < 0,$$

$$\text{as } \cos^2\alpha - \cos^2\beta > = < 0,$$

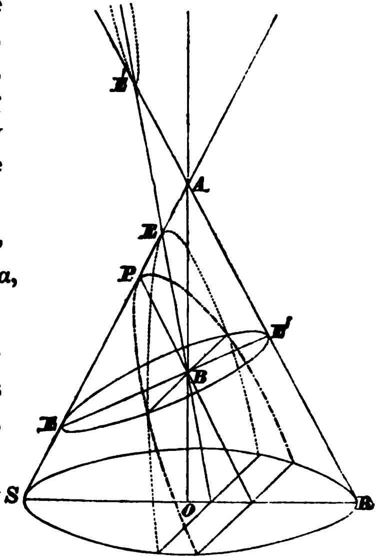
$$\text{as } \sin(\beta + \alpha) \,.\, \sin(\beta - \alpha) > = < 0,$$

$$\text{as } \sin(\beta - \alpha) > = < 0,$$

$$\text{or as } \beta > = < \alpha,$$

since $\beta + \alpha$ is by construction less than π, and therefore sin $(\beta + \alpha)$ always positive.

274. We may easily identify the above results with the forms of the curves that we have already discovered; for, let *SAR* be a cone, *AO* the axis, *B* the point where the cutting plane *EE′* cuts the axis; draw *BP* parallel to *AR*, then we have by our assumptions



$$AB = \alpha,\ \angle EAB = \beta,\ \angle EBA = \alpha,$$

$$\angle SAR = 2\beta,\ \angle AEB = \pi - \beta - \alpha,$$

and hence

$$\angle SAR + \angle AEB = \pi + \beta - \alpha.$$

If $\beta < \alpha$, these two angles are less than two right angles, and the point *E* will lie below the point *P*, and the section will evidently be limited in every direction, and be an ellipse.

If $\beta = \alpha$, the two angles = two right angles; the point *E* coincides with *P*, and the cutting plane is parallel to *AR*. The section will evidently be limited at *P* and unlimited in the direction *PB*, and be a parabola.

If $\beta > \alpha$, the two angles are greater than two right angles; the point *E* lies above *P*, and the cutting plane meets both sheets of the cone. The section will be unlimited in every direction, and be an Hyperbola.

275. It is from this property of the cone that the three curves derive their names. The Ellipse is so called because the two angles mentioned above fall short of (ἐλλείπειν) two right angles; the Hyperbola, because they exceed (ὑπερβάλλειν) two right angles; and the Parabola, because its axis is parallel (παραβάλλειν) to the side of the cone.

APPENDIX.

Properties of Quadratic Equations.

1. THE most general form of a quadratic is

$$ax^2 + bx + c = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or $$ax^2 + bx \quad = - c. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Multiply every term of (2) by $4a$ and add b^2 to each side, then

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

whence $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The roots of (1) are, therefore,

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}; \quad . \quad (5)$$

hence (i) if $b^2 > 4ac$, we shall have $b^2 - 4ac$ a *positive* quantity, and therefore $\sqrt{b^2 - 4ac}$ a *possible* quantity; and since in one root it is taken with +, and in the other with −, the two roots will be *real and different.*

(ii) If $b^2 = 4ac$, $\sqrt{b^2 - 4ac} = 0$, and therefore the two roots will be *real and equal.*

(iii) If $b^2 < 4ac$, $b^2 - 4ac$ is a *negative* quantity, and $\sqrt{b^2 - 4ac}$ is impossible, and therefore the two roots are impossible.

Hence, the roots of equation (1) are *real and different*, *real and equal*, or *impossible*, according as

$$b^2 > = < 4ac.$$

2. If α, β represent the roots of $ax^2 + bx + c = 0$, then

$$-\frac{b}{a} = \alpha + \beta, \quad \frac{c}{a} = \alpha\beta.$$

The values of α and β are given above (5); hence, adding and multiplying these values, we have

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}.$$

This property is expressed by saying that, if any quadratic be so written that the coefficient of x^2 is unity, then the coefficient of x is equal to *minus* the *sum* of the roots, and the last term is equal to the *product* of the roots.

3. If α, β be the roots of $ax^2 + bx + c$, then

$$ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

$$\text{For, } x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - (\alpha + \beta)x + \alpha\beta,$$

$$= (x - \alpha)(x - \beta);$$

$$\therefore\ ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

This may easily be shewn to be true for equations of all dimensions.

Cor. If one of the roots be 0, the corresponding factor is $x - 0$ or x; hence, when the factor x occurs in every term of an equation, it may be struck out, but it must be remembered that one root of that equation is $x = 0$.

4. *If there be two equations given of the second degree between two unknowns, the elimination of either of them will lead, generally, to an equation of the fourth degree.*

Let the equations be

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad . \quad . \quad (1)$$

$$A'x^2 + B'xy + C'y^2 + D'x + E'y + F' = 0, \quad . \quad . \quad (2)$$

which may be written

$$Px^2 + Qx + R = 0, \quad . \quad . \quad . \quad . \quad . \quad (3) \quad \times P$$

$$P'x^2 + Q'x + R' = 0. \quad . \quad . \quad . \quad . \quad . \quad (4) \quad \times P'$$

Eliminating x^2 between (3) and (4), we have

$$(P'Q - PQ')x + P'R - PR' = 0. \quad . \quad . \quad . \quad (5)$$

Eliminating R and R', and dividing by x, we have

$$(P'R - PR')x + Q'R - QR' = 0. \quad . \quad . \quad (6)$$

Eliminating x from (5) and (6) we have

$$(P'R - PR')^2 + (PQ' - P'Q)\,(Q'R - QR') = 0, \quad . \quad (7)$$

which manifestly contains y^4, since R and R' contain y^2. This is only a particular case of a general theorem, proved in most treatises on the *Theory of Equations*, namely that elimination between two equations of the m^{th} and n^{th} degrees, respectively, will produce an equation of the mn^{th} degree.

ANSWERS TO THE EXAMPLES.

I.

2. $\left(1, -\frac{3}{2}\right), \left(-\frac{1}{2}, 0\right), \left(\frac{1}{2}, -\frac{3}{2}\right)$ are the middle points.

3. $x = -\frac{9}{4}, \; y = 1.$ 4. $5, \sqrt{37}.$

5. $3\sqrt{2}, \; 3\sqrt{3}.$

II.

2. $y - 6x + 7 = 0.$

3. $(2y - y_2 - y_1)(y_3 - y_2) = (2x - x_1 - x_2)(x_3 - x_2)$ is one equation.

4. $y - x - 7 = 0.$

5. (i) $y + x - 6\sqrt{2} = 0$; (ii) $y + x + 6\sqrt{2} = 0.$

6. $x = \frac{3}{7}, \; y = \frac{1}{7}.$ 7. $x = \frac{17}{7}, \; y = \frac{24}{7}.$ 8. $y = mx + c.$

9. There are eight straight lines which fulfil the conditions of the problem.

10. $(a^2 + b^2)^{\frac{1}{2}}, \; \frac{(a^2 + b^2)^{\frac{3}{2}}}{ab}.$

11. Area $= \frac{b^2}{2} \cdot \frac{\sin(\alpha_1 - \alpha)\cos^2\alpha_2}{\sin(\alpha - \alpha_2)\sin(\alpha_1 - \alpha_2)}.$

12. $x + y = a + b.$

13. $(a' - a)y - (b' - b)x = a'b - ab'.$
$(a - a')y + (b - b')x = ab - a'b'.$

14. $x = \sqrt{3}y.$ 15. $x = 2, \; y = 3, \; x = 3, \; y = 4.$

16. $\frac{6}{\sqrt{13}}$. 17. $y + 2x = 0$.

18. $x + y = 2a$, or $= 4a$, according to the side on which the line is drawn.

20. $m = 3$.

26. $x = y = \frac{a}{4}\sqrt{6}$.

28. Co-ordinates of vertex being o, y', the intersection is o, $\frac{1}{3}y'$.

III.

1. $11y - (8 \pm 5\sqrt{3})x - 5(5 \mp \sqrt{3}) = 0$.

2. (i) $x = c$, $y = 0$; (ii) 135^0.

3. $(a \pm b)y = (b \mp a)(x - c)$.

4. $5y - 8x - 40 = 0$. 5. $\frac{25}{\sqrt{629}}$. 6. 90^0.

7. $y = (1 \pm \sqrt{2})(x + 2)$.

9. $\frac{13y - 12}{13x - 30} = \frac{23 \pm 5\sqrt{29}}{14}$. 13. Any angle.

14. $\frac{x}{a} + \frac{2y}{b} + 1 = 0$. $\frac{x}{a} - \frac{y}{2b} + 1 = 0$.

15. 45^0 or 135^0. 16. $2x = y$, $2y = x$.

17. $\frac{62}{11\sqrt{26}}$. 18. $\sqrt{21}$.

19. $\rho = 2a$, $\theta = \frac{\pi}{2}$; the angle $= \frac{1}{3}\pi$.

IV.

6. The tangents of the angles are m', $-m$, $\frac{m - m'}{1 + mm'}$, and the sides are

$$AB = \frac{b}{m'}, \quad AC = \frac{b(1 + m'^2)^{\frac{1}{2}}}{m - m'}, \quad BC = \frac{m'b(1 + m^2)^{\frac{1}{2}}}{m(m - m')}.$$

11. A straight line perpendicular to the base.

V.

1. Co-ordinates of centre are 3 and -2; rad. $= 3$.

2. A circle whose rad. $= r$ and a tangent to it.

3. $x + y = 3$. 4. $x + y + 1 = 0$.

5. $(1 + a^2).(x^2 + y^2) - 2r(x + ay) = 0$.

6. $x^2 - ax + y^2 = r^2 - \frac{a^2}{2}$. 7. An impossible locus.

8. (i) $D^2 = 4AF$, (ii) $E^2 = 4AF$.

9. See Art. 87. 10. $\frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

11. $x^2 + y^2 = \frac{9}{5}$. 12. $\frac{r}{\cos\alpha}$, $\frac{r}{\sin\alpha}$.

13. $x\cos(\alpha' + \alpha'') + y\sin(\alpha' + \alpha'') = r\cos(\alpha'' - \alpha')$.

15. A circle. 16. $y = -\frac{1}{m}x$.

17. An arc of a circle.

18. A circle, whose diameter = the radius of the given circle.

22. The segment of a circle on the base which contains the given angle.

25. $\frac{h(x' - x'') + k(y' - y'')}{\{(x' - x'')^2 + (y' - y'')^2\}^{\frac{1}{2}}}$. 26. $x \pm 3y = 10$.

27. $y - b = m(x - a) \pm r(1 + m^2)^{\frac{1}{2}}$.

28. $x^2 + y^2 - x'x - y'y = 0$. 29. $4y - 3x = 0$.

30. $\{4\gamma^2 - 2(\alpha - \beta)^2\}^{\frac{1}{2}}$.

31. $2(\alpha - \alpha_1)x + 2(\beta - \beta_1)y + \alpha_1^2 + \beta_1^2 - \alpha^2 - \beta^2 = \gamma_1^2 - \gamma^2$.

33. $2\left(r^2 - \frac{a^2b^2}{a^2 + b^2}\right)^{\frac{1}{2}}$.

36. If r = radius, and p = perpendicular from centre on the given line, the point is on the perpendicular at a distance $= \pm\sqrt{p^2 - r^2}$ from the point where it meets the line.

37. $\pm(al + bm - \delta) = r(l^2 + m^2)^{\frac{1}{2}}$. 38. A straight line.

40. $r^2 = \dfrac{a^2 - 2ab\cos\omega + b^2}{4\sin^2\omega}$.

42. $x^2 + 8ax + y^2 - 6ay = 0$.

43. A circle described on the line joining the point and centre, as diameter.

44. A circle whose centre is on the line joining the given point with the centre of the given circle.

50. The triangle must be isosceles, and the ratio one of equality.

51. The triangle must be equilateral.

52. A circle.

54. $(x - a \pm b' \mp b)^2 + (y - b - b')^2 = (b' - b)^2$.

55. $x^2 - 9x + y^2 - 5y + 14 = 0$. 58. $y - x = 0$.

59. $x^2 + y^2 - 2r(x + y) + r^2 = 0$.

61. $(r \pm r')x \pm \{a^2 - (r \pm r')^2\}^{\frac{1}{2}} y = ar$.

62. Write $r = \dfrac{a + b \pm \sqrt{a^2 + b^2}}{2}$ in the result of 59.

63. $x^2 + y^2 - 2\{a + b \pm (2ab)^{\frac{1}{2}}\}(x + y) + \{a + b \pm (2ab)^{\frac{1}{2}}\}^2 = 0$.

67. If AE be initial line $AP = \rho$, $\angle PAE = \theta$, the polar equation to the locus is $\rho \sin\theta = a \sin(\theta + \alpha)$, where $a = AE$, $\alpha = \angle EAB$.

68. $\dfrac{x}{y} = \dfrac{OB - OB'}{OA - OA'}$.

69. The polar co-ordinates of the centre are $\rho = 2$, $\theta = \dfrac{\pi}{3}$, and rad. $= 3$.

70. A circle whose centre is the intersection of the diagonals.

72. A circle whose radius $= (n + 1)r$, if $CP = n \,.\, AC$.

73. If α be the given angle, centre origin, the equation is

$$y = \tan\left(\alpha - \frac{\pi}{4}\right)x + \frac{r}{\sqrt{2}}\sec\left(\alpha - \frac{\pi}{4}\right).$$

74. A circle whose centre is the fixed extremity of one of the lines.

76. A straight line perpendicular to the line joining the fixed point and the centre.

VII.

4. $\cos\theta = \dfrac{5\sqrt{5} - 3\sqrt{3}}{4\sqrt{5}}$. 7. $\cos\theta = \pm\dfrac{5\sqrt{17}}{27}$.

8. The extremity of the minor axis. 14. $\dfrac{a^2}{m^2} - \dfrac{b^2}{n^2} = 1$.

17. The extremities of the latus rectum; $\tan^{-1} \pm e$.

18. The equation $\pm 3y \pm \sqrt{5}x - 9a = 0$ will represent the four tangents.

21. $r = eb$. 22. The extremity of the latus rectum.

30. $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$.

37. Lat. rect. $= 8a$; $e = \sqrt{5}$.

39. $\left(\dfrac{bm}{y}\right)^2 + \left(\dfrac{an}{x}\right)^2 = (m+n)^2$.

40. An ellipse whose equation is $\dfrac{x^2}{a^2} + \dfrac{b^2y^2}{a^4} = 1$.

42. $\sqrt{\dfrac{2e}{1+e}}$.

46. (i) a tangent at the vertex A; (ii) an hyperbola whose equation is $(a+c)^2y^2 - b^2x^2 = -b^2c^2$, $2c$ being $= SH$; (iii) a tangent at the vertex A'.

VIII.

17. The equation to the locus is $\dfrac{b^2}{y^2} + \dfrac{a^2}{x^2} = \left(\dfrac{a^2 - b^2}{x^2 + y^2}\right)^2$.

27. The equation is $a^2y^2 + b^2x^2 = 2a^2b^2$.

28. If $2c$ be the length of the chord, the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + c^2 \cdot \frac{a^2y^2 + b^2x^2}{a^4y^2 + b^4x^2} = 1.$$

39. The equation to the locus is $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

IX.

2. $y = mx - 2dm - dm^3$.   3. $y_1 = d(\sqrt{5} - 2)^{\frac{1}{2}}$, $y_2 = d(\sqrt{5} + 2)^{\frac{1}{2}}$.

8. Taking the centre of the circle as origin, and the given diameter as axis of x, the latus rectum = twice the abscissa of the point of contact.

11. $x = \dfrac{ab' - a'b}{b - b'}$, $y = \frac{1}{2}(b + b')$.

12. A parabola whose equation is $y^2 = \pm(\mu^{\frac{1}{4}} \pm \mu^{-\frac{1}{4}})^2 dx$, taking the upper or lower signs according as the points of contact are on the same or opposite sides of the axis.

15. $y = \dfrac{2a'a''}{a' + a''}x + \dfrac{d(a' + a'')}{2a'a''}$.

16. Parabolas whose latera recta = half that of the original curve.

19. Distance $= 2\sqrt{2}d$.   21. $y^2 = \dfrac{4dx^2}{2d - x}$.

25. Area $= 2a^{\frac{3}{2}}(d^{\frac{1}{2}} \sim \delta^{\frac{1}{2}})$.

26. $2d$.   28. $\dfrac{d \cos^2 \phi}{\sin \phi}$, $\dfrac{d}{\sin \phi}$.

29. A parabola whose focus is at the centre of the given circle.

30. $d(y + x)^2 = (y^2 + x^2)(y - x)$.

31. If the focus be pole, the axis initial line, the polar equation is

$$\rho = 2d \cot \frac{\theta}{2}.$$

33. $y = mx + r\sqrt{1 + m^2}$, where $m = \pm\left(\dfrac{\sqrt{4d^2 + r^2} - r}{2r}\right)^{\frac{1}{2}}$.

37. A parabola of which AB, AC are tangents.

38. Latus rectum $= 2\rho(1 - \cos 2\phi)$, if ρ and ϕ be the given quantities; the vertex is at a distance from the focus $= \dfrac{\rho}{2}(1 - \cos 2\phi)$.

43. $y^2 = h(x - h)$, where h is any constant.

44. $y - mx + m^3d + 2md = 0$, where m is the tangent of the angle which the normal makes with the axis.

45. A parabola whose equation is $y^2 = d(x - 3d)$.

46. $x = 0$, $x = 2(c - 2d)$, $y = \pm\left(\frac{2d}{c - 2d}\right)^{\frac{1}{2}} x$.

51. A parabola whose equation is $y^2 = d(x - d)$.

53. $y = -2d$, $x = -3d$.

54. An extremity of the latus rectum.

56. If the given straight line be the axis of y, and a perpendicular from the given point $(= c)$ the axis of x, the locus will be a parabola whose equation is $y^2 = 2c\left(x - \frac{c}{2}\right)$.

57. $b^2 = -da$. 58. $a^2 = 2b^2$.

X.

7. $lmn = -1$.

8. The three sides of a triangle formed by joining the points where the bisectors of the angles meet the opposite sides.

10. $61y^2 - 17xy - 65x^2 + 36y + 174x - 151 = 0$.

15. $\frac{x^2}{aa'} \pm \frac{2xy}{\sqrt{aa'bb'}} + \frac{y^2}{bb'} - \left(\frac{1}{a} + \frac{1}{a'}\right)x - \left(\frac{1}{b} + \frac{1}{b'}\right)y + 1 = 0$.

22. Conics touching the sides of the triangle α, β, γ, (i) at their middle points, (ii) at the feet of the perpendiculars from the angles, and (iii) where the bisectors of the angles meet the sides.

24. $\frac{a^2 - b^2}{a}$.

FORMULÆ AND RESULTS TO BE COMMITTED TO MEMORY.

The numbers refer to the Articles.

6. $PR^2 = (x'' - x')^2 + (y'' - y')^2 + 2(x'' - x')(y'' - y')\cos\omega,$

$\text{or } = (x'' - x')^2 + (y'' - y')^2.$

24—28. $\frac{x}{a} + \frac{y}{b} = 1, \; x\cos\alpha + y\cos(\omega - \alpha) - p = 0,$

$x\cos\alpha + y\sin\alpha - p = 0, \; y = mx + b,$

$\frac{y - y'}{s} = \frac{x - x'}{c} = l, \; \frac{y - y'}{x - x'} = \frac{y'' - y'}{x'' - x'}.$

37—40. $t = \frac{m - m'}{1 + mm'}, \; m' = \frac{m \pm t}{1 \mp mt}$

$= m$ when $t = 0$, and $= -\frac{1}{m}$ when $t = \infty$.

42, 43. $l = \pm \frac{Ax + By + C}{\sqrt{A^2 + B^2}} = \pm(x\cos\alpha + y\sin\alpha - p).$

47. $\alpha \pm k\beta = 0, \; \alpha \pm \beta = 0.$ 51. $y = \rho\sin\theta, \; x = \rho\cos\theta.$

52. $\rho = p\sec(\theta - \alpha).$ 59. $x = X + x', \; y = Y + y'.$

60. $x = X\cos\alpha - Y\sin\alpha, \; y = Y\cos\alpha + X\sin\alpha.$

73. $(x - a)^2 + (y - b)^2 = r^2, \; (x - a)^2 + (y - b)^2 + 2(x - a)(y - b)\cos\omega = r^2.$

76. $x^2 + y^2 = r^2.$ 82. $xx' + yy' = r^2.$ 86. $y = mx \pm r\sqrt{1 + m^2}.$

100—120. $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$ The curve a hyperbola, parabola, or ellipse according as $B^2 - 4AC >=< 0$.

If A or C, or both, $=0$, curve a hyperbola; if $B = 0$ and A or $C = 0$, curve a parabola; if $B = 0$, the axes are, if rectangular, parallel to the axes of the curve; if D and $E = 0$, the centre is origin; if $F = 0$, the origin on the curve.

108. $2Ax' + By' + D = 0, \; 2Cy' + Bx' + E = 0.$

110. $\tan 2\theta = \frac{B}{A - C}.$

To change formulæ for the ellipse to those for the hyperbola, write $-b^2$ for b^2.

114. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ 133. $e^2 = \frac{a^2 - b^2}{a^2}.$ 135. $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.$

137. $y = mx \pm \sqrt{m^2a^2 + b^2}.$ 139. $\frac{x}{a} + \frac{y}{b} = 0, \; \frac{x}{a} - \frac{y}{b} = 0.$

146. $CG = e^2x'$, $CT = \dfrac{a^2}{x'}$. 151. $CS = CH = ae$, $x \pm \dfrac{a}{e} = 0$.

152. Ellipse $SP = a + ex'$, $HP = a - ex'$, $SP + HP = 2a$.

153. Hyperbola $SP = ex' + a$, $HP = ex' - a$, $SP - HP = 2a$.

156. Latus rectum $= \dfrac{2b^2}{a}$. 157. $\rho = \dfrac{a(1-e^2)}{1 - e\cos\theta}$.

169. $mm' = -\dfrac{b^2}{a^2}$. 174. $\dfrac{x}{a} = -\dfrac{y'}{b}$, $\dfrac{y}{b} = \dfrac{x'}{a}$.

176. $a'^2 + b'^2 = a^2 + b^2$. 180. $a'b' \sin\gamma = ab$.

189. $xy = \dfrac{a^2 + b^2}{4}$. 191. $xy' + yx' = \dfrac{a^2 + b^2}{2}$.

197. $y^2 = 4dx$. 198. $yy' = 2d(x + x')$.

201. $y = mx + \dfrac{d}{m}$. 209. $x + d = 0$. 210. $SP = d + x'$.

213. Latus Rectum $= 4d$. 214. $SP = ST = SG$.

215. $\rho = \dfrac{2d}{1 - \cos\theta}$. 221. $y = \dfrac{2d}{m}$.

233. $\lambda\alpha + \mu\beta + \nu\gamma = 0$ identically.

234—239. $S - kS' = 0$, $S - k\alpha\beta = 0$, $S - k\alpha^2 = 0$, $\alpha\gamma - k\beta\delta = 0$, $\alpha\gamma - k\beta^2 = 0$.

243. $\dfrac{l}{\alpha} + \dfrac{m}{\beta} + \dfrac{n}{\gamma} = 0$.

244. $\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$.

265. $\dfrac{A}{A'} = \dfrac{B}{B'} = \dfrac{C}{C'}$.

269. $\dfrac{B'^2 - 4A'C'}{(A' + C')^2} = \dfrac{B^2 - 4AC}{(A + C)^2}$.

⁂ The Student should learn these formulæ, and exercise himself frequently in *interpreting* them.

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